Notes for section 1.1 Jan 14, 2009

In the following, we use the following notations:
a. r_1 denotes the first row of the matrix, r_2 denotes the second row of the matrix and r_3 denotes the third row of the matrix.
b. $r_i := cr_j + r_i$ This means that we replace the i-th row of the matrix by adding c times the j-th row to i-th row of the matrix.
$r_i := cr_i$ This means that we replace the i-th row of the matrix by multiplying c times the i-th row to the i-th row of the matrix.
$r_i \leftrightarrow r_j$ This means that we interchange the i-th row and the j-th row.

1. Use elementary row operations to solve the following linear system.
$x_1 + 5x_2 + 2x_3 = 1$
$-2x_1 - 8x_2 + 2x_3 = 12$
$x_1 + 3x_2 - 3x_3 = -9$

Solution: Our goal is to eliminate as many variables as possible for each equation.

The augmented matrix of the above system is
$$\begin{bmatrix} 1 & 5 & 2 & 1 \\ -2 & -8 & 2 & 12 \\ 1 & 3 & -3 & -9 \end{bmatrix}.$$
First, we use first row to eliminate the x_1 variable in equation 2 and equation 3 by (a) replacing the second row of the matrix by multiplying 2 times the first row to the second row of the matrix so we can cancel the coefficient of the x_1 variable in the second equation
(b) replacing third row of the matrix by multiplying -1 times first row to the third row of the matrix so we can cancel the coefficient of the x_1 variable in the third equation
$$\begin{matrix} \\ 2r_1 \mapsto \\ \\ (-1)r_1 \mapsto \\ \\ \end{matrix} \begin{bmatrix} 1 & 5 & 2 & 1 \\ (2 & 10 & 4 & 2) \\ -2 & -8 & 2 & 12 \\ (-1 & -5 & -2 & -1) \\ 1 & 3 & -3 & -9 \end{bmatrix}$$
$$r_2 := 2r_1 + \widetilde{r_2, r_3} := (-1)r_1 + r_3 \qquad \begin{matrix} \\ 2r_1 + r_2 \mapsto \\ (-1)r_1 + r_3 \mapsto \end{matrix} \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & -2 & -5 & -10 \end{bmatrix}$$

Next, we use second row to eliminate the x_2 variable in equation 3

$r_3 := \widetilde{r_2} + r_3 \quad r_2 + r_3 \mapsto \begin{bmatrix} 1 & 5 & 2 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ Now the third row tells us that $x_3 = 4$.

Next, we use third row to eliminate the x_3 variable in equation 1 and equation 2

$$\begin{matrix} \\ -2r_3 \mapsto \\ \\ -6r_3 \mapsto \\ \\ \end{matrix} \begin{bmatrix} 1 & 5 & 2 & 1 \\ (0 & 0 & -2 & -8) \\ 0 & 2 & 6 & 14 \\ (0 & 0 & -6 & -24) \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$r_1 := -2r_3 + \widetilde{r_1}, r_2 := -6r_3 + r_2 \quad \begin{matrix} -2r_3 + r_1 \mapsto \\ -6r_3 + r_2 \mapsto \\ \\ \end{matrix} \begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 2 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Next multiply $\frac{1}{2}$ to the second row to get

$\widetilde{r_2} := \frac{1}{2}r_2 \quad \frac{1}{2}r_2 \mapsto \begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Next use the second row to eliminate the x_2 variable in the first equation.

$$\begin{matrix} \\ -5r_2 \mapsto \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 5 & 0 & -7 \\ (0 & -5 & 0 & 25) \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$r_1 := \widetilde{-5r_2} + r_1 \quad -5r_2 + r_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

So we have $x_1 = 18$, $x_2 = -5$ and $x_3 = 4$

2. Use elementary row operations to solve the following linear system.

   $-3x_2 + 5x_3 = 1$
   $2x_1 - 7x_2 + 3x_3 = -2$
   $x_1 - 5x_2 + 4x_3 = -3$

   Solution:

   The augmented matrix of the above system is

   $$\begin{bmatrix} 0 & -3 & 5 & 1 \\ 2 & -7 & 3 & -2 \\ 1 & -5 & 4 & -3 \end{bmatrix}.$$

   There is no x_1 variable in the first variable. So we interchange the first row and third row. The coefficient of x_1 is 1 in the third row. The coefficient of x_1 is 2 in the second row. So we choose the third row instead of the second row.

   $$\begin{bmatrix} 0 & -3 & 5 & 1 \\ 2 & -7 & 3 & -2 \\ 1 & -5 & 4 & -3 \end{bmatrix} \widetilde{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ 0 & -3 & 5 & 1 \end{bmatrix}$$

   Now we use first row to eliminate the x_1 variable in equation 2

   $$-2r_1 \mapsto \begin{bmatrix} 1 & -5 & 4 & -3 \\ (-2 & 10 & -8 & 6) \\ 2 & -7 & 3 & -2 \\ 0 & -3 & 5 & 1 \end{bmatrix}$$

   $r_2 := \widetilde{-2r_2} + r_1$

   $$-2r_2 + r_1 \mapsto \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 3 \\ 0 & -3 & 5 & 1 \end{bmatrix}$$

   Now we use second row to eliminate the x_2 variable in equation 3

   $r_3 := \widetilde{r_2} + r_3$

   $$r_2 + r_3 \mapsto \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

   Now the third row implies that $0 = 4$ which is impossible. So this linear system is inconsistent (no solution.)

3. Find an equation involving a, $b$and c that makes the following augmented matrix corresponding to a consistent system.
$\begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 2 & b \\ 2 & 5 & 5 & c \end{bmatrix}$.

Solution: First, we we use first row to eliminate the x_1 variable in second and third equation.

$$\begin{matrix} \\ -r_1 \mapsto \\ \\ -2r_1 \mapsto \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & a \\ (-1 & -1 & -1 & -a) \\ 1 & 2 & 2 & b \\ (-2 & -2 & -2 & -2a) \\ 2 & 5 & 5 & c \end{bmatrix}.$$

$r_2 := -r_1 + \widetilde{r_2, r_3} := -2r_1 + r_3 \qquad \begin{matrix} \\ -r_1 + r_2 \mapsto \\ -2r_1 + r_3 \mapsto \end{matrix} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 3 & 3 & c-2a \end{bmatrix}$

Now we use second row to eliminate the x_2 variable in equation 3

$$\begin{matrix} \\ \\ -3r_1 \mapsto \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ (0 & -3 & -3 & -3b+3a) \\ 0 & 3 & 3 & c-2a \end{bmatrix}$$

$r_3 := \widetilde{-3}r_2 + r_3$

$$\begin{matrix} \\ \\ -3r_1 + r_3 \mapsto \end{matrix} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-2a-3b+3a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & a-3b+c \end{bmatrix}$$

Now the third row implies that $0 = a-3b+c$. This system is consistent only if $a-3b+c=0$. Assume $a-3b+c=0$. The augmented matrix can be written as $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{matrix} \\ -r_2 \mapsto \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & a \\ (0 & -1 & -1 & -b+a) \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_1 := \widetilde{-r_2} + r_1 \quad \begin{matrix} -r_2 + r_1 \mapsto \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -b+2a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we have $x_1 = -b+2a$, $x_2+x_3 = b-a$. This implies that $x_1 = -b+2a$, $x_2 = b-a-x_3$ and x_3 is arbitrary. Therefore this system is consistent if $a-3b+c=0$.