Notes for the class on Feb 4, 2009

1 1.5 Solution Sets of Linear System

Definition 1 *A system of linear equations is called homogeneous if it's of the form* $Ax = 0$ *where* A *is a* $m \times n$ *matrix and* x *is a* $n \times 1$ *vector ad* 0 *is the zero vector in* R^m.

Remark: $Ax = 0$ always has at least one trivial solution, i.e. $x = 0$ (the zero vector in R^n). For a given homogeneous equation, we are more interested in the nontrivial solution.
The following Theorem is obvious.

Theorem 1 *The homogeneous equation* $Ax = 0$ *has a nontrivial solution if and only if the equation has at least one free variable.*

Let's look at the following examples.

Example 1 *Determine whether the following equation has a nontrivial solution.*
(A)

$$\begin{cases} x_1 + 2x_2 + 2x_3 = 0 \\ x_1 + 3x_2 + 3x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0. \end{cases} \tag{1}$$

(B) $Ax = 0$ *where* $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix}$

Solution: 1^0 The augmented matrix for (A) is $\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$. It is row equivalent to $\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ($r_2 := r_2 - r_1$ and $r_3 = r_3 - r_1$) $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ So it has only the trivial solution $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. Note that there is no free variable in this case.

2^0 The augmented matrix is $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & -3 & 0 \\ 4 & 3 & -5 & 0 \end{bmatrix} \sim (r_2 := r_2 - 2r_1,\, r_3 := r_3 - 4r_1)$
$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim (r_2 := -r_2, r_3 := r_3 - r_2) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - r_2)$
$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So x_3 is the free variable and $x_1 - 2x_3 = 0$, $x_2 + x_3 = 0$.

This gives $x_1 = 2x_3$, $x_2 = -x_3$ and $x = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (this is called the parametric vector form).

Next we will see the relation between the solution of homogeneous equation $Ax = 0$ and inhomogeneous equation $Ax = b$ where $b \neq 0$.

Example 2 *Find the solution of the following inhomogeneous equation.* $Ax = b$ *where* $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix}$ *and* $b = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$.

Solution: 2^0 The augmented matrix is $[A\ b] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & -3 & -1 \\ 4 & 3 & -5 & -1 \end{bmatrix}$

$\sim (r_2 := r_2 - 2r_1,\, r_3 := r_3 - 4r_1) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$

$\sim (r_2 := -r_2, r_3 := r_3 - r_2) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - r_2) \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So x_3 is the free variable and $x_1 - 2x_3 = -1$, $x_2 + x_3 = -1$. This gives $x_1 = -1 + 2x_3$, $x_2 = -1 - x_3$ and $x = \begin{bmatrix} -1 + 2x_3 \\ -1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (this is called the parametric vector form).

Remark: From this example and previous example, we can see that the row redution process to solve $Ax = b$ and $Ax = 0$ is the same.

We have the following Theorem.

Theorem 2 *Suppose $Ax = b$ is consistent. Then the solution set of $Ax = b$ is of the form $x = p + x_h$ where $Ap = b$ and x_h is any solution of the homogeneous equation $Ax = 0$*

2 1.7 Linear independence

Definition 2 *A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in R^m is said to be linearly independent if the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ (zero vector in R^m) has only trivial solution.*
A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in R^m is said to be linearly dependent if the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ (zero vector in R^m) has nontrivial solution.

Using the notation in previous equation, we can relate the problem of linear dependence to the solution of homogeneous equation. Recall that the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ is the same as the matrix equation $Ax = 0$ where $A = [v_1\ v_2\ \cdots\ v_p]$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

So we have the following result.

Theorem 3 *A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in R^m is linearly independent iff the matrix equation $Ax = 0$ has only trivial solution where $A = [v_1\ v_2\ \cdots\ v_p]$ iff the augmented matrix $[v_1\ v_2\ \cdots\ v_p\ 0]$ has only trivial solution.*
A set of vectors $\{v_1, v_2, \cdots, v_p\}$ in R^m is linearly dependent iff the matrix equation $Ax = 0$ has nontrivial solution where $A = [v_1\ v_2\ \cdots\ v_p]$ iff the augmented matrix $[v_1\ v_2\ \cdots\ v_p\ 0]$ has nontrivial solution.

Example 3 *Determine whether the following vectors are linearly independent.* $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix}.$

Solution: Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix}$. Consider the augmented matrix $[v_1\ v_2\ v_3\ 0] = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$. From Example 1(A) on page 1, we know this system has only trivial solution. So these three vectors are linearly independent.

Example 4 *Determine if the columns of the matrix form a linearly independent set.* $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix}$

Solution: Consider the augmented matrix The augmented matrix is$[A\ 0] =$
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & -3 & 0 \\ 4 & 3 & -5 & 0 \end{bmatrix} \sim (r_2 := r_2 - 2r_1,\ r_3 := r_3 - 4r_1) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim$$
$$(r_2 := -r_2,\ r_3 := r_3 - r_2) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - r_2) \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
So x_3 is the free variable and $x_1 - 2x_3 = 0$, $x_2 + x_3 = 0$. This gives $x_1 = 2x_3$, $x_2 = -x_3$ and $x = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. So this system has nontrivial solution. Thus the columns of A form a linearly dependent set.

Example 5 *Find the value of h for which the following vectors are linearly dependent.* $\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$.

Solution: Consider the augmented matrix $\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix}$
$$\sim (r_2 := r_2 + r_1) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim (r_3 := r_3 + 3r_1) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix}$$

$$\sim (r_2 := \tfrac{r_2}{2}) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim (r_3 := r_3 + 7r_2) \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

This system has nontrivial solution if $h + 10 = 0$, that is $h = -10$. So these three vectors are dependent if $h = -10$.