2.8 Subspace of R^n 2.9 Dimensions and Ranks
Notes for the class on Feb 25th and March 2nd 2009

1 The definition of Subspace

Before we give the general definition, we examine a simple example in detail.

Example 1 *Let H be the set consisting of all vectors in the planes $x - 3y + 2z = 0$, i.e $H = \{(x, y, z)|x - 3y + 2z = 0\}$. We can see that H is a plane goes thru the origin. This set H has the following properties:*
(a) The zero vector $(0, 0, 0)$ is in the set H.
(b) Suppose u and v are two vectors in H. Then $u + v$ is also in H.
(c) For each vector u in H. Then cu is also in H where c is a scalar.

This leads to the following definition of subspace

Definition 1 *A subspace of R^n is any set H in R^n that satisfies the following three properties.*
(a) The zero vector is in H.
(b) For each u and v in H, then $u + v$ is in H.
(c) For each u in H and each scalar c, the vector cu is in H.

Example 2 *Let H be the set consisting of all vectors in the line $y = x + 1$, i.e $H = \{(x, y)|y = x + 1\}$. Since the zero vector $(0, 0)$ is not in H, so H is not a subspace.*

Example 3 *For v_1, v_2 in R^n. Given any vector b in $Span\{v_1, v_2\}$. We can find two numbers α and β such that $b = \alpha v_1 + \beta v_2$.*
We will show that $Span\{v_1, v_2\}$ is a subspace.
We can verify that
(a) The zero vector 0 can be written as $0 = 0 \cdot v_1 + 0 \cdot v_2$. So the zero vector is in $Span\{v_1, v_2\}$.
(b) Suppose $u \in Span\{v_1, v_2\}$ and $v \in Span\{v_1, v_2\}$. Then we can find a_1, a_2, b_1 and b_2 such that $u = a_1v_1 + a_2v_2$ and $v = b_1v_1 + b_2v_2$. Then $u + v = a_1v_1 + a_2v_2 + b_1v_1 + b_2v_2 = a_1v_1 + b_1v_1 + a_2v_2 + b_2v_2 = (a_1 + b_1)v_1 + (a_2 + b_2)v_2$ So $u + v$ is in $Span\{v_1, v_2\}$.
(c) Suppose $u \in Span\{v_1, v_2\}$. Then we can find a_1 and a_2 such that $u =$

$a_1v_1 + a_2v_2$. *So* $cu = c(a_1v_1 + a_2v_2) = ca_1v_1 + ca_2v_2$ *which is also in* $Span\{v_1, v_2\}$.

Remark 1 *For* $v_1, v_2, \cdots, v_p$ *in* R^n. *Similarly, we can show that* $Span\{v_1, v_2, \cdots, v_p\}$ *is a subspace.*

2 Column space and null space

In the following, we will associate two subspaces with a matrix.

Definition 2 *The column space of a matrix* A *is the set* $Col(A)$ *of the span of the column vectors of* A. *Suppose* A *is a* $m \times n$ *matrix.* $Col(A)$ *is a subspace in* R^m.

Example 4 $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 1 & -1 \end{bmatrix}$.

Then $Col(A) = Span\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}\}$ *is a subspace in* R^3

It's obvious that we have the following result

Theorem 1 *Suppose* A *is a* $m \times n$ *matrix. A vector* b *is in* $Col(A)$ *iff the linear equation* $Ax = b$ *is consistent iff* $[A|b]$ *is consistent.*

Definition 3 *Suppose* A *is a* $m \times n$ *matrix. The null space of a matrix* A *is the set* $Nul(A)$ *of all solutions to the homogeneous equation* $Ax = 0$, *i.e.* $Nul(A) = \{x|Ax = 0\}$. *Note that* $Nul(A)$ *is subspace in* R^n.

Given a subspace. We are interested in finding the minimal set of elements to describe the subspace.

Definition 4 *A basis for a subspace* H *of* R^n *is a linearly independent set in* H *that spans* H.

Definition 5 *The dimension of a nonzero subspace* H, *denoted by dim* H, *is the number of vectors in any basis for* H. *The dimension of the zero subspace* $\{0\}$ *is defined to be zero.*

It's obvious that $dim(R^n) = n$. So we have following characterization of a basis for R^n.
A set of vectors $\{v_1, v_2, \cdots v_p\}$ in R^n forms a basis for R^n
iff (a) $p = n$ (b) $\{v_1, v_2, \cdots v_p\}$ is independent, this is the same as the fact that the augmented matrix $[A|0] = [a_1\ a_2\ \cdots a_p|0]$ has no free variable.
iff $A = [v_1, v_2, \cdots v_p]$ is an invertible $n \times n$ matrix
iff $A = [v_1, v_2, \cdots v_p]$ is an $n \times n$ matrix and the augmented matrix $[A|0] = [a_1\ a_2\ \cdots a_p|0]$ has no free variable.

Note that if $p > n$ then $\{v_1, v_2, \cdots v_p\}$ is linearly dependent in R^n.
If $p < n$ then $\{v_1, v_2, \cdots v_p\}$ can't span R^n.

Example 5 *Determine which sets in the following column vectors of each matrix are basis for R^2 or R^3.*

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}, \begin{bmatrix}2 & 1 & 4\\-2 & 1 & -6\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}2 & 1 & 4\\-2 & 1 & -8\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}2 & 1 & 4 & 1\\-2 & 1 & 8 & 2\\1 & 1 & 1 & 3\end{bmatrix} \begin{bmatrix}2 & 1\\-2 & 1\\1 & 1\end{bmatrix}$$

Solution: 1^0 Consider $det(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}) = 4 - 6 = -1 \neq 0$. So $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is invertible and its column vectors form a basis.
2^0 $det(\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}) = 4 - 4 = 0$. So $\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}$ is not invertible and its column vectors are not a basis.

3^0 Consider the augmented matrix $\left[\begin{array}{ccc|c}2 & 1 & 4 & 0\\-2 & 1 & -6 & 0\\1 & 1 & 1 & 0\end{array}\right]$. Now we perform row reduction.

Switch first and third row. $\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\-2 & 1 & -6 & 0\\2 & 1 & 4 & 0\end{array}\right]$

$r_2 := r_2 + 2r_1$, $r_3 := r_3 - 2r_1$ $\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 3 & -4 & 0\\0 & -1 & 2 & 0\end{array}\right]$

$r_2 := -r_3$, $r_3 := r_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$

$r_3 := r_3 + (-3)r_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

So there are no free variable. The column vectors of $\begin{bmatrix} 2 & 1 & 4 \\ -2 & 1 & -6 \\ 1 & 1 & 1 \end{bmatrix}$ form a basis in R^3.

4^0 Consider the augmented matrix $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ -2 & 1 & -8 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$. Now we perform row reduction.

Switch first and third row. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 1 & -8 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right]$

$r_2 := r_2 + 2r_1$, $r_3 := r_3 - 2r_1$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$

$r_2 := r_2/3$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$

$r_3 := r_3 + r_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So x_3 is a free variable. The column vectors of $\begin{bmatrix} 2 & 1 & 4 \\ -2 & 1 & -8 \\ 1 & 1 & 1 \end{bmatrix}$ don't form a basis in R^3.

5^0. $\begin{bmatrix} 2 & 1 & 4 & 1 \\ -2 & 1 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$ has 4 column vectors in R^3. So they are linearly dependent. It can't be a basis for R^3.

6^0 $\begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 1 & 1 \end{bmatrix}$ has only two vectors. It can't span R^3. So it can't be a basis for R^3.

Now we want to see how to find a basis for $Col(A)$ and $Nul(A)$.

Recall that we can use row reduction to write the solution of $Ax = 0$ in parametric vector form. The parametric vector form will give us a basis for $Nul(A)$.

Example 6 *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$. *Find bases for* $Col(A)$ *and* $Nul(A)$, *and then find the dimension of these subspace.*

Solution: 1^0 Recall that $Nul(A) = \{x|Ax = 0\}$. So we have to find the solution of the equation $Ax = 0$.

We consider the augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right]$.

$r_2 := r_2 + (-2)r_1$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -5 & 0 \end{array}\right]$

$r_2 := r_2/(-5)$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$r_1 := r_1 + (-2)r_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$.

So the solution of $Ax = 0$ is $x_1 + x_3 = 0$ and $x_2 + x_3 = 0$ and x_3 is free. Thus $x_1 = -x_3$, $x_2 = -x_3$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. So $\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \}$ is a basis for $Nul(A)$ and $dim(Nul(A)) = 1$ (we have only one vector in the basis of $Nul(A)$.)

2^0 From the basis of $Nul(A)$, we have $A \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0$. Write $A = [a_1 \ a_2 \ a_3]$.

So $= [a_1 \ a_2 \ a_3] \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0$ and $-a_1 - a_2 + a_3 = 0$. Thus $a_3 = a_1 + a_2$. Recall that $Col(A) = Span\{a_1, a_2, a_3\} = Span\{a_1, a_2, a_1 + a_2\} = Span\{a_1, a_2\}$. From the row reduction process, we also know that $\{a_1, a_2\}$ is linearly independent. So the first column and the second column of A form a basis for $Col(A)$. Hence $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$ is a basis for $Col(A)$ and $dim(Col(A)) = 2$.

Suppose A is a $m \times n$ matrix with $A = [a_1 \ a_2 \ \cdots a_n]$. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $Nul(A)$. By definition, we have $Ax = 0$, i.e

$[a_1 \ a_2 \ \cdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$. So $x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$. Thus any nonzero vector x in $Nul(A)$ gives us a linear relation of the column vectors. This implies that the column vector of A corresponding to the free variable can be written as a linear combination of the non-free column vectors. So The column vectors of A that corresponding to the basic variables (nonfree variables) form a basis of $Col(A)$.

To find the basis of the $Col(A)$ and $Nul(A)$.

(1) Row reduce $[A|0]$ to row reduced echelon form.

(2) Express the solution of $Ax = 0$ in parametric vector form. Then we can find the basis for $Nul(A)$.

(3) The column vectors of A that corresponding to the basic variables (nonfree variables) form a basis of $Col(A)$.

Example 7 *Let* $A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}$. *Find bases for* $Col(A)$ *and* $Nul(A)$, *and then find the dimension of these subspace.*

Solution: 1^0 *Recall that Nul(A) is the set of all solution of the homogeneous equation* $Ax = 0$. *Consider the augmented matrix* $\left[\begin{array}{cccc|c} -3 & 6 & -1 & 1 & 0 \\ 1 & -2 & 2 & 3 & 0 \\ 2 & -4 & 5 & 8 & 0 \end{array}\right]$.

First we switch the first and the second row to get $\left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ -3 & 6 & -1 & 1 & 0 \\ 2 & -4 & 5 & 8 & 0 \end{array}\right]$

$r_2 := r_2 + 3r_1 \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 5 & 10 & 0 \\ 2 & -4 & 5 & 8 & 0 \end{array}\right]$

$r_3 := r_3 + (-2r_1) \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 5 & 10 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$ $r_2 := r_2/5 \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$

$r_3 := r_3 - r_2 \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$r_1 := r_1 - 2r_2 \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

So x_2 *and* x_4 *are free variables. We have* $x_1 - 2x_2 - x_4 = 0$ *and* $x_3 + 2x_4 = 0$. *Hence* $x_1 = 2x_2 + x_4$ *and* $x_3 = -2x_4$.

The solution is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 0 \\ -2x_4 \\ x_4 \end{bmatrix}$

$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. *So the basis for* $Nul(A)$ *is* $\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \}$ *and* $dim(Nul(A)) = 2$.

2^0 *From the row reduction result, we know that* x_1 *and* x_3 *are not free veriables. So the first column and the third column of* A *form a basis for* $Col(A)$.

Hence $\{\begin{bmatrix}-3\\1\\2\end{bmatrix}$ *and* $\begin{bmatrix}-1\\2\\5\end{bmatrix}\}$ *form a basis for* $Col(A)$ *and* $dim(Col(A)) = 2$.

The basis for $Span\{v_1, v_2, \cdots, v_p\}$ is the same as the basis for $Col(A)$ where $A = [a_1 \quad \cdots \quad a_p]$

Example 8 *Find a basis for the subspace spanned by the given vectors* $\{\begin{bmatrix}-3\\1\\2\end{bmatrix}, \begin{bmatrix}6\\-2\\-4\end{bmatrix}, \begin{bmatrix}-1\\2\\5\end{bmatrix}, \begin{bmatrix}1\\3\\8\end{bmatrix}\}$.

Solution: Consider the matrix $A = \begin{bmatrix}-3 & 6 & -1 & 1\\1 & -2 & 2 & 3\\2 & -4 & 5 & 8\end{bmatrix}$. From previous question, we know that $\{\begin{bmatrix}-3\\1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\2\\5\end{bmatrix}\}$ form a basis for $Col(A)$. So $\{\begin{bmatrix}-3\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\5\end{bmatrix}\}$ is a basis for the subspace spanned by the given vectors $\{\begin{bmatrix}-3\\1\\2\end{bmatrix}, \begin{bmatrix}6\\-2\\-4\end{bmatrix}, \begin{bmatrix}-1\\2\\5\end{bmatrix}, \begin{bmatrix}1\\3\\8\end{bmatrix}\}$.

Theorem 2 *We denote* $rank(A) = dim(Col(A))$. *If* A *is a* $m \times n$ *matrix, then* $rank(A) + dim(Nul(A)) = n$.

Proof: Since A is a $m \times n$ matrix, there are n variable for the homogeneous equation $Ax = 0$.
The number of non-free variables + the number of free variables = n.
Recall that the number of non-free variables=dim(Col(A))=rank(A) and the number of free variables=dim (Nul(A)). Hence $rank(A) + dim(Nul(A)) = n$.

Definition 6 *Suppose the set* $\mathfrak{B} = \{u_1, u_2, \cdots u_p\}$ *is a basis for a subspace* H. *For each* b *in* H, *we can find* $x_1, x_2 \cdots, x_p$ *such that* $x_1u_1 + x_2u_2 + \cdots +$

$x_p u_p = b$. *The coordinate of b relative to the basis* $\mathfrak{B}$ *is the vector* $[b]_{\mathfrak{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

Theorem 3 *Suppose* $\mathfrak{B} = \{u_1, u_2, \cdots u_p\}$ *is a basis for a subspace* H. *Let* $A = [u_1 \ u_2 \ \cdots \ u_p]$. *Then* $A[b]_{\mathfrak{B}} = b$.

To find coordinate of a vector b, just solve $Ax = b$. Then $[b]_{\mathfrak{B}} = x$.

Example 9 *Find the* $\mathfrak{B}$*-coordinate of the of the vector* $b = \begin{bmatrix} 22 \\ 0 \\ 14 \end{bmatrix}$ *relative to the basis* $\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} \right\}$

Solution: Let $A = \begin{bmatrix} -3 & 7 \\ 1 & 5 \\ -4 & -6 \end{bmatrix}$. We solve $Ax = \begin{bmatrix} 22 \\ 0 \\ 14 \end{bmatrix}$.

Consider the augmented matrix $[A|b] = \begin{bmatrix} -3 & 7 & 22 \\ 1 & 5 & 0 \\ -4 & -6 & 14 \end{bmatrix}$.

Switch first and second row. $\begin{bmatrix} 1 & 5 & 0 \\ -3 & 7 & 22 \\ -4 & -6 & 14 \end{bmatrix}$

$r_2 := r_2 + 3r_1$, $r_3 := r_3 + 4r_1$, $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 22 & 22 \\ 0 & 14 & 14 \end{bmatrix}$

$r_2 := r_2/22$, $r_3 := r_3/14$, $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$r_3 := r_3 - r_2$ $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $r_1 := r_1 - 5r_2$ $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So $x_1 = -5$, $x_2 = 1$ and the coordinate of $[b]_{\mathfrak{B}} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.