2.2 The inverse of a Matrix (Notes for the class on Feb 23, 2009)

1 The definition of inverse Matrix

The inverse of a real number $a \neq 0$ is a real number denoted a^{-1} such that $a^{-1}a = 1$.

In the following, we use I to denote the $n \times n$ identity matrix. Recall that $AI = IA = A$ for any $n \times n$ matrix A.

We can define the inverse of a $n \times n$ matrix by the following.

Definition 1 *The inverse of an $n \times n$ matrix A is a $n \times n$ matrix such that $AB = I$ and $BA = I$ where I is the $n \times n$ identity matrix. The matrix A is said to be invertible if it has an inverse. The notation for the inverse of A is A^{-1}, i.e. $AA^{-1} = A^{-1}A = I$.*

Example 1 *Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.
(a) Show that $A^2 - 2A = 3I$.
(b) Use the formula $A^2 - 2A = 3I$ to find an formula for A^{-1} in terms of A.
Solution: 1^0 *First, we compute*

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2 \cdot 2 & 1 \cdot 2 + 2 \cdot 1 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

So $A^2 - 2A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5-2 & 4-4 \\ 4-3 & 5-2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I$.

2^0 *First, note that $A^2 - 2A = A \cdot A - 2A \cdot I = A(A - 2I)$. Here we have used the fact that $A = AI$. So $A^2 - 2A = 3I$ implies that $A(A - 2I) = 3I$, $\frac{1}{3}A(A - 2I) = I$ and $A \cdot \underbrace{(\frac{1}{3}(A - 2I))}_{A^{-1}} = I$. Hence $A^{-1} = \frac{1}{3}(A - 2I)$.*

Example 2 *Let* $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. *It can be shown that $A^3 - 3A^2 - 9A = 5I$.*

What's the formula for A^{-1}?
Solution: Note that $A^3 - 3A^2 - 9A = A(A^2 - 2A - 9I)$. From $A^3 - 3A^2 - 9A = 5I$, we get $A(A^2 - 3A - 9I) = 5I$ and $A(\underbrace{\frac{1}{5}(A^2 - 3A - 9I)}_{A^{-1}}) = I$ So

$A^{-1} = \frac{1}{5}(A^2 - 3A - 9I)$

2 The inverse of a 2×2 matrix

Next we will see how to find the inverse of a 2 matrix. Recall that the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $det(A) = ad - bc$. We can find the inverse of a 2×2 matrix quite easily by hand. Here is the theorem

Theorem 1 *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.*
If $det(A) = ad - bc = 0$ then A is not invertible.
If $det(A) = ad - bc \neq 0$. Then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof: It can be verified by direct computation.

Example 3 *Find the inverse of the following matrices if possible.*
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$

Solution: 1^0 $det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$. So the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.
2^0 $det(\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}) = 2 \cdot 2 - (-1)\cot(-4) = 4 - 4 = 0$. So the matrix $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ is not invertible.

3 The solution of $Ax = b$ and the inverse matrix

Suppose A is a $n \times n$ matrix and A is invertible. Now we can use it to solve the linear equation $Ax = b$. Multiply A^{-1} to both sides of the equation, we get $A^{-1}Ax = A^{-1}b$. Now use the fact that $A^{-1}A = I$ and $Ix = x$. We get $x = A^{-1}b$. So we have the following theorem.

Theorem 2 *Suppose A is invertible. For any $b \in R^n$, the linear equation $Ax = b$ has a unique solution $x = A^{-1}b$. In particular, $Ax = 0$ has only trivial solution if A is invertible.*

Next, we will see how to use row reduction to find the inverse of a matrix. Let's look at a 3×3 example. Suppose the inverse of A exists. We write the inverse $A^{-1} = [u_1\ u_2\ \ u_3]$ where u_1, u_2 and u_3 are the column vectors of A^{-1} Recall that the 3×3 identity matrix is $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then we have $AA^{-1} = A[u_1\ u_2\ u_3] = [Au_1\ Au_2\ Au_n]$. So $AA^{-1} = I$ is the same as $[Au_1\ Au_2\ Au_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This implies that

$$Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } Au_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So we can find u_1 and u_2 and u_3 by solving the linear system corresponding the augmented matrix $\begin{bmatrix} & 1 \\ A & 0 \\ & 0 \end{bmatrix}$, $\begin{bmatrix} & 0 \\ A & 1 \\ & 0 \end{bmatrix}$ and $\begin{bmatrix} & 0 \\ A & 0 \\ & 1 \end{bmatrix}$. This is the same as considering the augmented matrix $[A|I]$. If $[A|I]$ is row equivalent to $I|B$ then $B = A^{-1}$.

Here is the method to compute the inverse of A.

First, consider the augmented matrix $[A|I]$. Then we perform roe reduction on $[A|I]$. There are two cases.

Case 1. $[A|I]$ is row equivalent to $[I|A^{-1}]$.

Case 2. There is at least one free variable. Then A is not invertible.

Example 4 *Find the inverse of the following matrix if possible*
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 4 & 11 & -5 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 6 & 1 \\ -1 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix}.$$

Solution: (a) Consider the augmented matrix $\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

switch 1st row and 2nd row. $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$r_3 := r_3 + (-1)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$

$r_3 := r_3 + (-1)r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$

$r_3 := -r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

$r_2 := r_2 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

$r_1 := r_1 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

So the inverse of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(b) Consider the augmented matrix $\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 4 & 11 & -5 & 0 & 0 & 1 \end{array}\right]$

Switch first row and second row to get $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 11 & -5 & 0 & 0 & 1 \end{array}\right]$

$r_2 := r_2 + (-2)r_1 \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 5 & -3 & 1 & -2 & 0 \\ 4 & 11 & -5 & 0 & 0 & 1 \end{array}\right]$

$r_3 := r_3 - 4r_1 \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 5 & -3 & 1 & -2 & 0 \\ 0 & 15 & -9 & 0 & -4 & 1 \end{array}\right]$

$r_3 := r_3 + (-3)r_2 \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 5 & -3 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 2 & 1 \end{array}\right]$ So there is at least one free variable. Therefore the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 4 & 11 & -5 \end{bmatrix}$ is not invertible.

(c) Consider the augmented matrix $\left[\begin{array}{ccc|ccc} 3 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$

First, we switch 1st row and second row to get $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$

$$r_2 := r_2 + (-3)r_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & -4 & 1 & -3 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$r_3 := r_3 + (-1)r_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & -4 & 1 & -3 & 0 \\ 0 & 2 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$r_3 := r_3 - r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & -4 & 1 & -3 & 0 \\ 0 & 0 & 3 & -1 & 2 & 1 \end{array}\right]$$

$$r_3 = \tfrac{1}{3}r_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & -4 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1/3 & 2/3 & 1/3 \end{array}\right]$$

$$r_2 = r_2 + 4r_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & 1/3 \end{array}\right]$$

$$r_1 = r_1 + (-2)r_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -2/3 \\ 0 & 2 & 0 & -1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & 1/3 \end{array}\right]$$

$$r_2 := \tfrac{1}{2}r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -2/3 \\ 0 & 1 & 0 & -1/6 & -1/6 & 2/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & 1/3 \end{array}\right].$$

So the inverse of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 2/3 & -1/3 & -2/3 \\ -1/6 & -1/6 & 2/3 \\ -1/3 & 2/3 & 1/3 \end{bmatrix}$

(d) Consider the augmented matrix $\left[\begin{array}{ccc|ccc} 2 & 6 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$

switch 1st and 3rd row $\left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 6 & 1 & 1 & 0 & 0 \end{array}\right]$

$r_2 := r_2 + r_1$ $\left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 & 1 \\ 2 & 6 & 1 & 1 & 0 & 0 \end{array}\right]$

$r_3 := r_3 + (-2)r_1$ $\left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 & 1 \\ 0 & -2 & -1 & 1 & 0 & -2 \end{array}\right]$

$r_2 := r_2 + r_3$ $\left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & -2 & -1 & 1 & 0 & -2 \end{array}\right]$

$r_3 := r_3 + 2r_2$ $\left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 3 & 2 & -4 \end{array}\right]$

$r_3 := -r_3$ $\left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -3 & -2 & 4 \end{array}\right]$

$r_1 := r_1 - r_3$ $\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 3 & 2 & -3 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -3 & -2 & 4 \end{array}\right]$

$$r_1 := r_1 + (-4)r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -3 & -2 & 4 \end{array}\right]$$

So the inverse of $\begin{bmatrix} 2 & 6 & 1 \\ -1 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & -2 & 4 \end{bmatrix}$ Recall that if A is invertible then $Ax = b$ has a unique solution $x = A^{-1}b$.

Example 5 *Use the formula of A^{-1} to solve $Ax = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ where $A = \begin{bmatrix} 2 & 6 & 1 \\ -1 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$.*

Solution: In previous question, we have found $A^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & -2 & 4 \end{bmatrix}$. So

$$x = A^{-1}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \cdot 2 + (-2) \cdot (-1) + 1 \cdot 4 \\ 1 \cdot 2 + 1 \cdot (-1) + (-1) \cdot 4 \\ -3 \cdot 2 + (-2) \cdot (-1) + 4 \cdot 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 12 \end{bmatrix}.$$