Notes for the class on Feb 18, 2009

1 2.1 Matrix Algebra

Definition 1 *An $m \times n$ matrix is a collection of numbers (called entries) $[a_{ij}]_{1\le i\le m, 1\le j\le n}$. We use the notation A_{ij} to denote the (i,j)-th entry a_{ij}.*

$$A = \begin{bmatrix} & & i-th\ row & & \\ a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ & & \vdots & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ & & \vdots & & \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} j-th\ column.$$

We can define the addition of two matrices and the scalar multiplication by the following.

Definition 2 *Let A and B are two matrices of the size. Then $(A+B)_{ij} = A_{ij} + B_{ij}$ and $(cA)_{ij} = cA_{ij}$*

Example 1 *Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$. Find $2A - 3B$.*

Solution:

$$\begin{aligned} & 2A - 3B \\ =& 2\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \end{bmatrix} - 3\begin{bmatrix} 0 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix} \\ =& \begin{bmatrix} 2 & 2 & -2 \\ 4 & 2 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -6 \\ 6 & -9 & 6 \end{bmatrix} \\ =& \begin{bmatrix} 2+0 & 2+3 & -2-6 \\ 4+6 & 2-9 & -6+6 \end{bmatrix} \\ =& \begin{bmatrix} 2 & 5 & -8 \\ 10 & -7 & 0 \end{bmatrix} \end{aligned}$$

Now we want to define the multiplication of two matrices.

Definition 3 *(row column rule) Let A be a $m \times n$ matrix and and B be a $n \times p$ matrix. Then AB is a $m \times p$ matrix where*

$(AB)_{ij} =$ *the dot product of the* $i-th$ *row of* A *and the* $j-th$ *column of the matrix* B.

Example 2 *Let* $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find AB.
(b) Find BA
(b) Find B^2.

Solution: 1^0 Note that A is a 2×3 matrix and B is a 3×3 matrix. So AB is a 2×3 matrix. We have

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 4 + 1 \cdot 0 + (-1) \cdot 1 & 1 \cdot (-2) + 1 \cdot 1 + (-1) \cdot 0 & 1 \cdot (-2) + 1 \cdot 0 + (-1) \cdot 1 \\ 2 \cdot 4 + 1 \cdot 0 + (-3) \cdot 1 & 2 \cdot (-2) + 1 \cdot 1 + (-3) \cdot 0 & 2 \cdot (-2) + 1 \cdot 0 + (-3) \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -3 \\ 5 & -3 & -7 \end{bmatrix}.$$

2^0 Since B is a 3×3 matrix and A is a 2×3 matrix, BA is not defined.

3^0 $B^2 = \begin{bmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4\cdot 4+(-2)\cdot 0+(-2)\cdot 1 & 4\cdot(-2)+(-2)\cdot 1+(-2)\cdot 0 & 4\cdot(-2)+(-2)\cdot 0+(-2)\cdot 1 \\ 0\cdot 4+1\cdot 0+0\cdot 1 & 0\cdot(-2)+1\cdot 1+0\cdot 0 & 0\cdot(-2)+1\cdot 0+0\cdot 1 \\ 1\cdot 4+0\cdot 0+1\cdot 1 & 1\cdot(-2)+0\cdot 1+1\cdot 0 & 1\cdot(-2)+0\cdot 0+1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -10 & -10 \\ 0 & 1 & 0 \\ 5 & -2 & -1 \end{bmatrix}.$$

Definition 4 *The $n \times n$ identity matrix is denoted by I_n where the diagonal entries are all one and the rest of entries are zero, i.e.* $I = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$.
Sometime we will just use I to denote the identity matrix.

Example 3 *The 2×2 identity matrix is* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
The 3×3 identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 1 *We have the following properties of matrix. The following matrices has the right sizes for which the indicated products and sums are defined*
(a) $A(BC) = (AB)C$
(b) $A(B + C) = AB + AC$
(c) $(B + C)A = BA + CA$
(d) $c(AB) = (cA)B = A(cB)$ *(here c is a number).*
(e) $I_m A = A = AI_n$ *(A is a $m \times n$ matrix).*

Matrix multiplication is different from the scalar multiplication. In general, $AB \neq BA$. The following is an example.

Example 4 *Let* $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
Then $AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ *and* $BA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.
So $AB \neq BA$.

Also $AB = 0$. we cannot conclude $A = 0$ or $B = 0$. The following is an example.

Example 5 *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \\ 0 & 0 \end{bmatrix}$.

Nest, we define the transport of a matrix.

Definition 5 *The transport of a* $m \times n$ *matrix is the* $n \times mn$ *matrix, denoted by* A^T*, whose columns are formed by the corresponding rows of* A

Example 6 *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. *Then* $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Example 7 *Let* $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Then* $v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. *Compute* uv^T *and* $u^T v$.

Solution: $uv^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$.

$u^T v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 + 10 + 18 \end{bmatrix} = [32]$.