Notes for the class on Feb 11, 2009

1 1.8 Linear Transformation

Recall that if $A = [a_1 \ a_2 \ \cdots \ a_n]$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \ x_n \end{bmatrix}$ then

$Ax = x_1a_1 + x_2a_2 + \cdots + x_na_n$.

Definition 1 *Given a $m \times n$ matrix. We can define a transformation $T : R^n \mapsto R^m$ by $T(x) = Ax$ for $x \in R^n$. The set R^n is called the domain of T and $T(x)$ is called the image of of T. The set of all images $T(x)$ is called the range of T.*

Remark:
$b \in R^m$ is in the range of T
iff there exists x in R^n such that $Ax = b$
iff there exists x in R^n such that $x_1a_1 + x_2a_2 + \cdots + x_na_n = b$
iff b is a linear combination of the columns of A
iff b is in the span of columns of A
iff the linear system $[A \ b]$ is consistent

Let's look at the following examples.

Example 1 *Let* $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix}$.
(A) Find the transformation T associated with the matrix A.
(B) Is $b = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ in the range of T? Also find x such that $Ax = b$. Also find x such that $Ax = b$. Determine whether x is unique.
(C) Is $c = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$ in the range of T? Also find x such that $Ax = c$. Is there more than one x whose image is under T is b?

Solution: 1^0 $T(x) = Ax = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 4 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix} =$ $\begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + x_2 - 3x_3 \\ -3x_1 + 3x_2 - 5x_3 \end{bmatrix}$.

2^0 $b = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ in the range of T

iff there exists x such that $T(x) = b$

iff there exists x such that $Ax = b$

Consider augmented matrix for $[A\ b] == \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & -3 & -1 \\ 4 & 3 & -5 & -1 \end{bmatrix}$

$\sim (r_2 := r_2 - 2r_1,\ r_3 := r_3 - 4r_1) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$

$\sim (r_2 := -r_2,\ r_3 := r_3 - r_2) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - r_2) \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So x_3 is the free variable and $x_1 - 2x_3 = -1$, $x_2 + x_3 = -1$. This gives $x_1 = -1 + 2x_3$, $x_2 = -1 - x_3$ and $x = \begin{bmatrix} -1 + 2x_3 \\ -1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (this is called the parametric vector form). There are infinitely many x such that $Ax = b$ (so it is not unique), i.e. $T(x) = b$.

3^0 Consider augmented matrix for $[A\ c] = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & -3 & 3 \\ 4 & 3 & -5 & 8 \end{bmatrix}$

$\sim (r_2 := r_2 - 2r_1,\ r_3 := r_3 - 4r_1) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

$\sim (r_2 := -r_2,\ r_3 := r_3 - r_2) \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The last row implies that $0 = 1$ which is impossible. So this system is inconsistent and c is not in the range of T.

Example 2 *Let* $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$. *Let* $T(x) = Ax$. *Is* $b = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ *in the range of* T*? Also find* x *such that* $Ax = b$*. Determine whether* x *is unique.*

Solution:

1^0 Consider augmented matrix for $[A\ b] == \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 4 & 3 & -1 \end{bmatrix}$

$\sim (r_2 := r_2 - 2r_1,\ r_3 := r_3 - 4r_1) \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$\sim (r_2 := -r_2,\ r_3 := r_3 - r_2) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim (r_1 := r_1 - r_2) \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This gives $x_1 = -1$, $x_2 = -1$ and $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. So we have a unique $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ such that $Ax = b$, i.e. $T(x) = b$.

Given a matrix A. The transformation $T(x) = Ax$ satisfies $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$. (because $A(u+v) = Au + Av$ and $A(cu) = cAu$.) This leads to the following definition.

Definition 2 *A transformation* T *is called linear if* $T(u+v) = T(u) + T(v)$ *and* $T(cu) = cT(u)$

Note that if T is a linear transformation then $T(au+bv) = T(au) + T(bv) = aT(u) + bT(v)$.

Example 3 *Let* $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *Suppose* T *is a linear transformation with* $T(e_1) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ *and* $T(e_2) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. *Find* Tx *where* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: Note that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2$. Since T is linear, we have

$T(x) = T(x_1e_1 + x_2e_2) = x_1T(e_1) + x_2T(e_2) = x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 - x_2 \\ -2x_1 + 2x_2 \end{bmatrix}.$

Example 4 *Let* $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *Suppose* T *is a linear transformation with* $T(e_1 + e_2) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ *and* $T(2e_1 + 3e_2) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. *Find* $T(x)$ *where* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: Note that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1e_1 + x_2e_2$. Since T is linear, we have
$T(x) = T(x_1e_1 + x_2e_2) = x_1T(e_1) + x_2T(e_2)$. It suffices to find $T(e_1)$ and $T(e_2)$. Using $T(e_1 + e_2) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $T(2e_1 + 3e_2) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, we have $T(e_1) + T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $2T(e_1) + 3T(e_2) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Multiply (-2) to $T(e_1) + T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and add it to $2T(e_1) + 3T(e_2) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, we get $-2T(e_1) - 2T(e_2) + 2T(e_1) + 3T(e_2) = -2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. So $T(e_2) = -2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$. Using $T(e_1) + T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, we get $T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - T(e_1) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -8 \end{bmatrix}$ Thus $T(x) = x_1T(e_1) + x_2T(e_2) =$

$$x_1 \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} x_2 \\ -3x_1 + 4x_2 \\ 6x_1 - 8x_2 \end{bmatrix}.$$