Notes for the class on Jan 26, 2009

The following is an example that the matrix is in row echelon form:

$$\begin{bmatrix} 1 & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \textit{pivot vector} & \textit{pivot vector} & & & \textit{pivot vector} & \textit{pivot vector} & \end{bmatrix}.$$

We can use the pivot vector to eliminate the entries above it to get the following reduced echelon form. Also the non pivot vectors corresponds to free variables.
Also the pivot vectors corresponds to basic variables.

$$\begin{bmatrix} 1 & 0 & * & * & 0 & 0 & * \\ 0 & 1 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & free\ variable & free\ variable & & & \end{bmatrix}.$$

One can express the basic variables as free variables from the row reduced echelon form.

Example:

Use elementary row operations to solve the following linear system.

$$\begin{aligned} x_3 - x_4 - x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 &= 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 &= 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 &= 6 \end{aligned}$$

Solution:

The augmented matrix of the above system is

$$\begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{bmatrix}.$$

First, we use interchange row one and row two so we have a nonzero pivot.

$$\begin{bmatrix} 2 & 4 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{bmatrix}$$

Next we divide row one by 2 to get $\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{bmatrix}$.

Next we eliminate all other entries in the pivot column.

$$\begin{array}{r} \\ \\ (-2)r_1 \mapsto \\ \\ (-3)r_1 \mapsto \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ (-2 & -4 & -2 & -4 & -2 & -4) \\ 2 & 4 & 3 & 3 & 3 & 4 \\ (-3 & -6 & -3 & -6 & -3 & -6) \\ 3 & 6 & 6 & 3 & 6 & 6 \end{bmatrix}.$$

$$\begin{array}{r} \\ \\ (-2)r_1 + r_3 \mapsto \\ (-3)r_1 + r_4 \mapsto \end{array} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{bmatrix}.$$

$$\begin{array}{r} \\ \\ -r_2 \mapsto \\ \\ -3r_2 \mapsto \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ (0 & 0 & -1 & 1 & 1 & -4) \\ 0 & 0 & 1 & -1 & 1 & 0 \\ (0 & 0 & -3 & 3 & 3 & -12) \\ 0 & 0 & 3 & -3 & 3 & 0 \end{bmatrix}.$$

$$\begin{array}{r} \\ \\ -r_2 + r_3 \mapsto \\ (-3)r_1 + r_4 \mapsto \end{array} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{bmatrix}.$$

$$\frac{r_3}{2} \mapsto \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{bmatrix}.$$

$$-6r_3 + r_4 \mapsto \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \textit{pivot vector} & & \textit{pivot vector} & & \textit{pivot vector} & \end{bmatrix}.$$

$$\begin{matrix} \\ -r_4 \mapsto \\ \\ r_4 \mapsto \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ (0 & 0 & 0 & 0 & -1 & 2) \\ 0 & 0 & 1 & -1 & -1 & 4 \\ (0 & 0 & 0 & 0 & 1 & -2) \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{matrix} r_1 - r_4 \mapsto \\ r_2 + r_4 \mapsto \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{matrix} \\ -r_2 \mapsto \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 4 \\ (0 & 0 & -1 & 1 & 0 & -2) \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{matrix} -r_2 + r_1 \mapsto \\ \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & x_2 \; \textit{free variable} & x_3 & x_4 \; \textit{free variable} & x_5 & \end{bmatrix}.$$

So we know that x_2 and x_4 are free variables.

From the row reduced echelon form, we have $x_1 + 2x_2 + 3x_4 = 2$, $x_3 - x_4 = 2$ and $x_5 = -3$. This implies that $x_1 = 4 - 2x_2 - 3x_4$, $x_3 = 2 + x_4$ and $x_5 = -3$.

Thus the general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2x_2 - 3x_4 \\ x_2 \\ 2 + x_4 \\ x_4 \\ -2 \end{bmatrix}$

(Theorem)
A linear system is inconsistent if and only if the reduced echelon form has row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & b \end{bmatrix}$ $b \neq 0$.
A linear system is consistent if and only if the reduced echelon form has no row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & b \end{bmatrix}$ $b \neq 0$.
If the system is consistent then there is only one solution if no free variables.
If the system is consistent then there is infinitely many solutions if we have free variables.