**Notes for 6.1, 6.2 and 6.3**

# 1 6.1

**Definition 1.1** *The inner product or dot product between two vectors* $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}$ *and* $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots y_n \end{bmatrix}$ *is*

$$x \cdot y = x^T y = \begin{bmatrix} x_1 & x_2 & \cdots x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

*The length of the vector is* $||x|| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
*The distance between two vectors is*

$$dist(x, y) = ||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

We have the following properties of the dot product and the length.

**Theorem 1.1** *(a)*$x \cdot y = y \cdot x$.
*(b)* $x \cdot x > 0$ *if* $x \neq 0$.
*(c)* $x \cdot (y + z) = x \cdot y + x \cdot z$.
*(d)* $||ax|| = |a| ||x||$ *where* $a$ *is a number and* $x \in R^n$.

**Example 1** *Let* $u = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ *and* $v = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. *Find* $u \cdot v$, $||u||$, $||v||^2$ *and* $||u+v||^2$.

Solution: Compute $u \cdot v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 2 - 2 + 2 = 2$, $||u|| = \sqrt{u \cdot u} = \sqrt{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} = \sqrt{4 + 1 + 1} = \sqrt{6}$. $||v||^2 = v \cdot v = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 1 + 4 + 4 =$

9. To find $||u+v||^2$, we first compute $u+v = \begin{bmatrix}2\\1\\1\end{bmatrix} + \begin{bmatrix}1\\-2\\2\end{bmatrix} = \begin{bmatrix}2+1\\1-2\\1+2\end{bmatrix} = \begin{bmatrix}3\\-1\\3\end{bmatrix}$. Then $||u+v||^2 = (u+v)\cdot(u+v) = \begin{bmatrix}3\\-1\\3\end{bmatrix} \cdot \begin{bmatrix}3\\-1\\3\end{bmatrix} = 9+1+9 = 19$.

**Definition 1.2** *Two vectors are called orthogonal if $x \cdot y = 0$.*

From Pythahorean Theorem, we know that the $x$ is perpendicular to the vector $y$ iff $||x||^2 + ||y||^2 = ||x+y||^2$. We can simplify $||x+y||^2 = (x+y)\cdot(x+y) = x\cdot(x+y) + y\cdot(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y = ||x||^2 + ||y||^2 + 2x\cdot y$ (use $x\cdot y = y\cdot x$). Thus $||x||^2 + ||y||^2 = ||x+y||^2$ is the same as $||x||^2 + ||y||^2 = ||x||^2 + ||y||^2 + 2x\cdot y$. This implies that $2x\cdot y = 0$ and $x\cdot y = 0$. Thus we have the following theorem

**Theorem 1.2** *$x$ is perpendicular to the vector $y$ iff $x$ is orthogonal to the vector $y$ iff $x\cdot y = 0$.*

**Example 2** *Determine which pairs of vectors are orthogonal.*
*1.* $a = \begin{bmatrix}2\\1\end{bmatrix}$ *and* $b = \begin{bmatrix}-1\\1\end{bmatrix}$.
*2.* $u = \begin{bmatrix}2\\1\\2\end{bmatrix}$ *and* $v = \begin{bmatrix}-3\\2\\2\end{bmatrix}$.

Solution: $1^0$ Compute $a\cdot b = \begin{bmatrix}2\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\end{bmatrix} = -2+1 = -1 \neq 0$. So $a$ is not orthogonal to $b$.
$1^0$ Compute $u\cdot v = \begin{bmatrix}2\\1\\2\end{bmatrix} \cdot \begin{bmatrix}-3\\2\\2\end{bmatrix} = -6+2+4 = 0 \neq 0$. So $u$ is orthogonal to $v$.

Next we define the notion of orthogonal complement.

**Definition 1.3** *The set of all vectors $z$ that are orthogonal to a subspace $W$ in $R^n$ is called the orthogonal complement of $W$ and is denoted by $W^\perp$,*

$$W^\perp = \{z \in R^n | z\cdot y = 0, \ \textit{for every } y \in W\}$$

.

**Example 3** *What is $W^\perp$ if $W = Span\{u_1, u_2, \cdots, u_p\}$?*

Solution: $W^\perp = \{z \in R^n | z \cdot u_1 = 0, z \cdot u_2 = 0, \cdots, z \cdot u_p = 0\}$.

**Example 4**

*Let* $W = Span\{\begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\2\end{bmatrix}\}$. *Describe the subspace $W^\perp$ and find a basis for $W^\perp$.*

Solution: $W^\perp = \{x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in R^3 | x \cdot \begin{bmatrix}2\\2\\1\end{bmatrix} = 0, z \cdot \begin{bmatrix}1\\-2\\2\end{bmatrix} = 0\}$

$= \{x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in R^3 | 2x_1 + 2x_2 + x_3 = 0, x_1 - 2x_2 + 2x_3 = 0\}$.

Consider the matrix $A = \begin{bmatrix}2 & 2 & 1\\1 & -2 & 2\end{bmatrix}$. So $W^\perp = Nul(A)$. $\begin{bmatrix}2 & 2 & 1\\1 & -2 & 2\end{bmatrix} \widetilde{r_1 \leftrightarrow r_2} \begin{bmatrix}1 & -2 & 2\\2 & 2 & 1\end{bmatrix}$ $\widetilde{r_2 := r_2 + (-2)r_1} \begin{bmatrix}1 & -2 & 2\\0 & 6 & -3\end{bmatrix} \widetilde{r_2 := r_2/6} \begin{bmatrix}1 & -2 & 2\\0 & 1 & -1/2\end{bmatrix}$ $\widetilde{r_1 := r_1 + 2r_2} \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1/2\end{bmatrix}$. So $x_1 + x_3 = 0$ and $x_2 - x_3/2 = 0$. So $x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-x_1\\x_3/2\\x_3\end{bmatrix} = x_3 \begin{bmatrix}-1\\1/2\\1\end{bmatrix}$. So $\begin{bmatrix}-1\\1/2\\1\end{bmatrix}$ is a basis for $W^\perp$.

# 2 6.2 Orthogonal sets

**Definition 2.1** *An set of nonzero vectors $\{u_1, u_2, \cdots, u_p\}$ is called an orthogonal set if $u_i \cdot u_j = 0$ whenever $1 \leq i \neq j \leq p$.*

**Definition 2.2** *An orthogonal basis $\{u_1, u_2, \cdots, u_p\}$ for a subspace $W$ is a basis that is also orthogonal, i.e. $u_i \cdot u_j = 0$ whenever $1 \leq i \neq j \leq p$.*

**Theorem 2.1** *If $\mathfrak{B} = \{u_1, u_2, \cdots, u_p\}$ is an orthogonal basis for a subspace $W$ and $y \in W$, then $y = c_1 u_1 + \cdots + c_p u_p$ where $c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1}$, $c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2}$ , $\cdots$,*

$c_p = \frac{y \cdot u_p}{u_p \cdot u_p}$. *This implies that the coordinate vector* $[y]_{\mathfrak{B}} = \begin{bmatrix} \frac{y \cdot u_1}{u_1 \cdot u_1} \\ \frac{y \cdot u_2}{u_2 \cdot u_2} \\ \vdots \\ \frac{y \cdot u_p}{u_p \cdot u_p} \end{bmatrix}$.

**Example 5** *Show that* $\{u_1, u_2, u_3\}$ *is an orthogonal basis for* $R^3$ *where* $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ *and* $u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. *Also express* $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$ *as a linear combination of* $\{u_1, u_2, u_3\}$.

Solution: $1^0$ We need to compute $u_1 \cdot u_2$, $u_1 \cdot u_3$ and $u_2 \cdot u_3$.
Compute $u_1 \cdot u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$, $u_1 \cdot u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 0 - 2 = 0$ and $u_2 \cdot u_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0$. So we have $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 0$ and $u_2 \cdot u_3 = 0$. Hence $\{u_1, u_2, u_3\}$ is an orthogonal basis for $R^3$.
$2^0$ From previous Theorem, we know that $x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3$.
Now we compute $x \cdot u_1 = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 8 + 0 - 3 = 5$, $u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 + 0 + 1 = 2$, $x \cdot u_2 = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -8 - 16 - 3 = -27$, $u_2 \cdot u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = 1 + 16 + 1 = 18$, $x \cdot u_3 = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 16 - 4 + 6 = 18$ and $u_3 \cdot u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 4 + 1 + 4 = 9$. Using $x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3$, we have $x = \frac{5}{2} u_1 + \frac{-27}{18} u_2 + \frac{18}{9} u_3 = \frac{5}{2} u_1 - \frac{3}{2} u_2 + 2 u_3$.

Remark: One can verify that

$$\begin{aligned}
&\frac{5}{2}u_1 - \frac{3}{2}u_2 + 2u_3 \\
=&\frac{5}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{3}{2}\begin{bmatrix}-1\\4\\1\end{bmatrix} + 2\begin{bmatrix}2\\1\\-2\end{bmatrix} = \begin{bmatrix}\frac{5}{2}+\frac{3}{2}+4\\0-6+2\\\frac{5}{2}-\frac{3}{2}-4\end{bmatrix} = \begin{bmatrix}8\\-4\\-3\end{bmatrix} \qquad (2.1)\\
=&x.
\end{aligned}$$

**Definition 2.3** *An set of nonzero vectors $\{u_1, u_2, \cdots, u_p\}$ is called an orthonormal set if*
*(a) it is an orthogonal set, i.e. $u_i \cdot u_j = 0$ whenever $1 \leq i \neq j \leq p$.*
*(b) each $u_i$ ia a unit vector, i.e.$||u_1|| = ||u_2|| = \cdots = ||u_p|| = 1$.*

Remark: Given am orthogonal set $\{u_1, u_2, \cdots, u_p\}$. Then we can normalize it to get an orthonormal set $\{\frac{u_1}{||u_1||}, \frac{u_2}{||u_2||}, \cdots, \frac{u_p}{||u_p||}\}$.

**Example 6** *Normalize the following orthogonal set to get an orthonormal set.* $u_1 = \begin{bmatrix}3\\4\end{bmatrix}$, $u_2 = \begin{bmatrix}4\\-3\end{bmatrix}$.

Solution: We can verify that $u_1 \cdot u_2 = 0$. Compute $||u_1|| = \sqrt{3^2 + 4^2} = 5$ and $||u_2|| = \sqrt{4^2 + (-3)^2} = 5$.
So $\{\frac{u_1}{||u_1||}, \frac{u_2}{||u_2||}\} = \{\frac{u_1}{5}, \frac{u_2}{5}\} = \{\begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\end{bmatrix}, \begin{bmatrix}\frac{4}{5}\\-\frac{3}{5}\end{bmatrix}\}$ is an orthonormal set.

# 3 6.3 Orthogonal projection

Given a vector $y \in R^3$ and a plane $W$ thru the origin. We can decompose the vector $y$ into two vectors $\widehat{y}$ and $z$ such that $\widehat{y}$ is the projection of $y$ onto the subspace $W$ and $z \in W^\perp$. This can be generalized to the following orthogonal projection Theorem. First, we define the notion of orthogonal projection.

**Definition 3.1** *Suppose $W = Span\{u_1, u_2, \cdots, u_p\}$ and $\{u_1, u_2, \cdots, u_p\}$ is an orthogonal basis for the subspace $W$. Then given any vector $y \in R^n$. The vector $\widehat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p}u_p$ is called the orthogonal projection of $y$ onto the subspace $W$. We also write $\widehat{y} = Proj_W(y)$.*

Note that a vector $y \in W$ iff $y = Proj_W(y)$.

**Theorem 3.1** *(Orthogonal projection Theorem) Suppose* $W = Span\{u_1, u_2, \cdots, u_p\}$ *and* $\{u_1, u_2, \cdots, u_p\}$ *is an orthogonal basis for the subspace* $W$*. Any vector* $y \in R^n$ *can be written uniquely as* $y = Proj_W(y) + z$ *where* $Proj_W(y) = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 + \cdots + \frac{y\cdot u_p}{u_p\cdot u_p}u_p \in W$ *and* $z = y - Proj_W(y) \in W^\perp$*.*

Remark. To use this theorem, we need to make sure that $\{u_1, u_2, \cdots, u_p\}$ is an orthogonal basis for the subspace $W$.

**Example 7** *Let* $W = Span\{u_1, u_2\}$ *where* $u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ *and* $u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$*. Write* $y = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ *as a sum of a vector in* $W$ *and a vector orthogonal to* $W$*.*

Solution. First, we compute $u_1 \cdot u_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = 5 + 3 - 8 = 0.$ So $\{u_1, u_2\}$ ia an orthogonal basis for the subspace $W$. By the Orthogonal projection Theorem, we can write $y = Proj_W(y) + z$ where $Proj_W(y) = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 \in W$ and $z = y - Proj_W(y) \in W^\perp$. We need to compute $y \cdot u_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = 2+9-10 = 1$, $u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = 1+9+4 = 14$, $y \cdot u_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = 10 + 3 + 20 = 33$, $u_2 \cdot u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = 25 + 1 + 16 = 42$. So $Proj_W y = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 = \frac{1}{14}u_1 + \frac{33}{42}u_2 = \frac{1}{14}u_1 + \frac{11}{14}u_2 = \frac{1}{14}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{11}{14}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} + \frac{55}{14} \\ \frac{3}{14} + \frac{11}{14} \\ \frac{-2}{14} + \frac{44}{14} \end{bmatrix} = \frac{3}{14} + \frac{11}{14} \begin{bmatrix} \frac{1}{14} + \frac{55}{14} \\ \frac{3}{14} + \frac{11}{14} \\ \frac{-2}{14} + \frac{44}{14} \end{bmatrix} \begin{bmatrix} \frac{56}{14} \\ \frac{14}{14} \\ \frac{42}{14} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and $z = y - Proj_W(y) = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$. So $y = Proj_W y + z$ where $Proj_W(y) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and $z = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} \in W^\perp$. We can verify that $Proj_W(y) \cdot z =$

$$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = -8 + 2 + 6 = 0.$$