Solution to Linear Algebra (Math 2890) Review Problems II

1. (a) What is a subspace in R^n?

   Solution: A subspace of R^n is any set H in R^n that satisfies the following three properties. (I) The zero vector is in H. (II) For each u and v in H, then $u + v$ is in H. (III) For each u in H and each scalar c, the vector cu is in H.

   (b) Is the set $\{(x, y, z)|x + y + z = 1\}$ a subspace?
   Solution: This is not a subspace since the zero vector $(0, 0, 0)$ is not in the set.

   (c) Is the set $\{(x, y, z)|x - y - z = 0, x + y - z = 0\}$ a subspace?
   Solution: Yes. This is a subspace. This can be regarded as the nullspace of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

   Here $Nul(A) = \{(x, y, z)| \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.

   (d) What is a basis for a subspace?
   Solution: A basis for a subspace H of R^n is a linearly independent set in H that spans H.

   (e) What is the dimension of a subspace?
   Solution:The dimension of a nonzero subspace H is the number of vectors in any basis for H.

   (f) What is the column space of a matrix?
   Solution: The column space of a matrix A is the set of the span of the column vectors of A.

   (g) What is the null space of a matrix?
   Solution:The null space of a matrix A is the set of all solutions to the homogeneous equation $Ax = 0$, i.e. $Nul(A) = \{x|Ax = 0\}$.

   (h) What is an eigenvalue of a matrix A?
   Solution: Let A be a $n \times n$ matrix. A scalar λ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.

(i) What is an eigenvector of a matrix A?
Solution: Let A be a $n \times n$ matrix. A scalar λ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.

(j) What is the characteristic polynomial of a matrix A ?
Solution: The polynomial $det(A - \lambda I)$ is the the characteristic polynomial of a matrix A.

(k) What is the subspace spanned by the vectors $v_1,\ v_2,\ \cdots,\ v_p$? Solution: The subspace spanned by $v_1,\ v_2,\ \cdots,\ v_p$ is the set of all possible linear combination of $v_1,\ v_2,\ \cdots,\ v_p$, i.e. $Span\{v_1,\ \cdots,\ v_n\} = \{c_1v_1 + c_2v_2 + \cdots + c_pv_p | c_1, c_2, \cdots, c_p \ are\ real\ numbers\}$

2. Find the inverses of the following matrices if they exist.

$$A = \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Solution: (a) Since $det(A) = -1$, we have $A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & -7 \end{bmatrix}$

(b)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \widetilde{r_1 \longleftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$r_2 := r_2 + \widetilde{(-1)r_1, r_3} := r_3 + (-1)r_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$r_3 \widetilde{:= r_3} + r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & 1 & -2 \end{array}\right]$$

$$\widetilde{r_3 :=} \frac{r_3}{2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right]$$

$$r_2 := \widetilde{r_2 + (}-1)r_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right]$$

So $B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$.

(c)

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 5 & 6 & 7 & 0 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right] \overset{r_2 := r_2 - 2r_1}{\sim} \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_2 \longleftrightarrow r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_2 := r_2 - 2r_1, r_3 := r_3 - 8r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 9 & 18 & 16 & -8 & 1 \end{array}\right]$$

$$\overset{r_3 := r_3 + (-3)r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 0 & 0 & 7 & -5 & 1 \end{array}\right]$$

So C only has one free variable (or two pivot vectors) and C is not invertible.

(d) $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\overset{r_3 := r_3 + (-2)r_1, r_4 := r_4 + (-1)r_1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_3 := r_3 + (-3)r_2, r_4 := r_4 + (-2)r_2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 & 0 & 1 \end{array}\right]$$ So

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}.$$

3. (a) Let A be an 3×3 matrix. Suppose $A^3 + 2A^2 - 3A + 4I = 0$. Is A invertible? Express A^{-1} in terms of A if possible.
Solution: From $A^3+2A^2-3A+4I = 0$, we have $A^3+2A^2-3A = -4I$, $A(A^2 + 2A - 3I) = -4I$ and $A \cdot (-\frac{1}{4}(A^2 + 2A - 3I)) = I$. So $A^{-1} = -\frac{1}{4}(A^2 + 2A - 3I)$.
(b) Suppose $A^3 = 0$. Is A invertible?
Solution: If A is invertible then $A^{-2}A^3 = A^{-2}0$ and $A = 0$ which is not invertible. So A is not invertible.

4. Describe the values of t so that the following matrices are invertible

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & t+1 & 3 \\ 1 & t & t+1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 0 & t \\ -1 & 0 & t & 0 \\ 0 & -t & 0 & 1 \\ -t & 0 & -1 & 0 \end{bmatrix}$$

Solution:

(a)

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & t+1 & 3 \\ 1 & t & t+1 \end{bmatrix} \overset{r_2 := r_2 + (-1)r_1,\, r_3 := r_3 + (-1)r_1}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & t & 1 \\ 0 & t-1 & t-1 \end{bmatrix}$$

$$\overset{r_3 := \frac{1}{t-1}r_3 \ if\ t-1 \neq 0,\, r_2 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & t & 1 \end{bmatrix} \overset{r_3 := r_3 + (-t)r_2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-t \end{bmatrix}$$

Thus M is invertible if $t \neq 1$.

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 & t \\ -1 & 0 & t & 0 \\ 0 & -t & 0 & 1 \\ -t & 0 & -1 & 0 \end{bmatrix} \underbrace{}{interchange\ 1st\ and\ 2nd\ row, r_1 := (-1)r_1} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & -t & 0 & 1 \\ -t & 0 & -1 & 0 \end{bmatrix}$$

$$\widetilde{r_4 := r_4 + t \cdot r_1} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & -t & 0 & 1 \\ 0 & 0 & -1-t^2 & 0 \end{bmatrix} \widetilde{r_3 := r_3 + t \cdot r_2} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1+t^2 \\ 0 & 0 & -1-t^2 & 0 \end{bmatrix}$$

$$\widetilde{r_3 := \frac{1}{1+t^2} r_3, r_4 := \frac{-1}{1+t^2} r_4, r_3 \leftrightarrow r_4} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus A has no free variable (or four pivot vectors) and A is invertible for all t. Note that we have used the fact that $1 + t^2 \neq 0$ in the computation.

5. Find all values of a and b so that the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ -a \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is two-dimensional.

Solution: Consider the matrix $A = \begin{bmatrix} 0 & b & -2 \\ 1 & 1 & 2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$$\widetilde{interchange\ first\ row\ and\ second\ row} \begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\widetilde{r_4 := r_1 + r_4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$\widetilde{interchange\ second\ row\ and\ forth\ row}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$

$\widetilde{divide\ second\ row\ by\ 2}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$ $\widetilde{r_3 := r_3 + ar_2, r_4 := r_4 - br_2}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & a \\ 0 & 0 & -2-b \end{bmatrix}$.

Now the first and second vectors are pivot vectors. So $rank(A) = 2$ if $a = 0$ and $-2 - b = 0$.

So $a = 0$ and $b = -2$

6. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \}$. You can assume that $\mathcal{B}$ is a basis for R^3

(a) Which vector x has the coordinate vector $[x]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. So $x = A[x]_B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-3+0 \\ 0-2+0 \\ 0-1+4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$

(b) Find the β-coordinate vector of $y = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

Solution. We have to solve $Ax = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right] \widetilde{r_2 := \tfrac{1}{2} r_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 3 \end{array}\right] \widetilde{r_2 := r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\widetilde{r_3 := \tfrac{1}{2}r_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \widetilde{r_1 := r_1 - 3r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

So $[y]_B = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$.

7. Let

$$M = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

Find bases for $Col(M)$ and $Nul(M)$, and then state the dimensions of these subspaces

Solution: $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \widetilde{r_2 := -r_1 + r_2, r_3 := -r_2 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}$

$\widetilde{r_3 := -2r_2 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \widetilde{r_1 := -2r_2 + r_3} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$ is a basis for $Col(A)$ and $dim(Col(A)) = 2$.

The solution to $Mx = 0$ is $x_1 + x_3 - x_4 = 0$ and $x_2 + 2x_3 + x_4 = 0$. So $x = \begin{bmatrix} -x_3 + x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Hence the basis for $Nul(M)$ is $\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \}$ and $dim(Nul(M)) = 2$.

8. Find a basis for the subspace spanned by the following vectors $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \}$.

What is the dimension of the subspace?

Solution: Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$r_2 := r_2 - \widetilde{r_1, r_3} := r_3 - r_1 \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -3 \\ 0 & -1 & -1 \end{bmatrix}$

$r_2 :\widetilde{= r_2}/(-3) \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} r_3 :\widetilde{= r_3} + r_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

So the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \}$ is a basis.

The dimension of the subspace is 2.

9. Determine which sets in the following are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify your answer

(a) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. Solution: Since $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, the set $\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \}$ is dependent. It is not a basis.

(b) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Yes. This set forms a basis since they are independent and span R^3.

(c) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

This is not a basis since it doesn't span R^3.

(d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This set forms a basis since they are independent and span R^3

(e) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. This is not a basis since it is dependent.

10. Let A be the matrix
$$A = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues and a basis of eigenvectors for A.

(c) Diagonalize the matrix A if possible.

(d) Find a polynomial $P(A)$ in A such that $P(A) = 0$. Verify your answer.

(e) Find the formula for A^k where k is an positive integer. Solution:

1. $A - \lambda I = \begin{bmatrix} -3-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$.
So $det(A - \lambda I) = (-3-\lambda)(3-\lambda) - 16 = \lambda^2 - 9 - 16 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$
So the characteristic equation is $(\lambda - 5)(\lambda + 5) = 0$.

2. Solving the characteristic equation $(\lambda - 5)(\lambda + 5) = 0$, we get that the eigenvalues are $\lambda = 5$ and $\lambda = -5$.

3. When $\lambda = 5$, we have
$A - \lambda I = \begin{bmatrix} -3-5 & -4 \\ -4 & 3-5 \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -4 & 2 \end{bmatrix} \; \widetilde{r_2 := r_2 - r_1/2} = \begin{bmatrix} -8 & -4 \\ 0 & 0 \end{bmatrix}$
$\widetilde{r_1 := -r_1/8} = \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$.

The solution of $(A - 5I)x = 0$ is $x_1 + x_2/2 = 0$, i.e. and $x_1 = -x_2/2$
So $Null(A - I) = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2/2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\}$.
We can choose $x_2 = 2$ to get $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ which is a basis for the eigenspace corresponding to eigenvalue 5.

4. When $\lambda = -5$, we have
$A - \lambda I = \begin{bmatrix} -3+5 & -4 \\ -4 & 3+5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \; \widetilde{r_1 := r_1/2} \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix}$
$\widetilde{r_2 := r_2 + 4r_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

The solution of $(A + 5I)x = 0$ is $x_1 - 2x_2 = 0$, i.e. and $x_1 = 2x_2$
So $Null(A - I) = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue -5 is $\{\begin{bmatrix}2\\1\end{bmatrix}\}$.

Let $P = \begin{bmatrix}-1 & 2\\2 & 1\end{bmatrix}$. Then $P^{-1} = \frac{1}{-5}\begin{bmatrix}1 & -2\\-2 & -1\end{bmatrix} = \begin{bmatrix}-\frac{1}{5} & \frac{2}{5}\\\frac{2}{5} & \frac{1}{5}\end{bmatrix}$.

So $A = \begin{bmatrix}-1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}5 & 0\\0 & -5\end{bmatrix}\begin{bmatrix}-\frac{1}{5} & \frac{2}{5}\\\frac{2}{5} & \frac{1}{5}\end{bmatrix}$

and $A^k = \begin{bmatrix}-1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}5^k & 0\\0 & (-5)^k\end{bmatrix}\begin{bmatrix}-\frac{1}{5} & \frac{2}{5}\\\frac{2}{5} & \frac{1}{5}\end{bmatrix}$.

Let $P(\lambda) = (\lambda-5)(\lambda+5) = \lambda^2 - 25$. Let $P(A) = A^2 - 25I$. Then $A^2 - 25I = 0$. One can verify this by checking $A^2 = \begin{bmatrix}-3 & -4\\-4 & 3\end{bmatrix}\begin{bmatrix}-3 & -4\\-4 & 3\end{bmatrix} = \begin{bmatrix}25 & 0\\0 & 25\end{bmatrix} = 25\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 25I$. Hence $A^2 - 25I = 0$.

11. Let A be the matrix $\begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$.

(a) Prove that $det(A-\lambda I) = -(\lambda-1)^2(\lambda-4)$.
(b) Find the eigenvalues and a basis of eigenvectors for A.
(c) Diagonalize the matrix A if possible.
(d) Find the matrix exponential e^A. Solution.

a. 1. $A-\lambda I = \begin{bmatrix}2-\lambda & 1 & 1\\1 & 2-\lambda & 1\\1 & 1 & 2-\lambda\end{bmatrix}$.

So $det(A-\lambda I) = (2-\lambda)^3 + 1 + 1 - (2-\lambda) - (2-\lambda) - (2-\lambda) = (4-4\lambda+\lambda^2)(2-\lambda) + 2 - 6 + 3\lambda = 8 - 8\lambda + 2\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 4 + 3\lambda$ $= -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda-1)^2(\lambda-4)$. So the characteristic equation is $-(\lambda-1)^2(\lambda-4) = 0$.

2. Solving the characteristic equation $-(\lambda-1)^2(\lambda-4) = 0$, we get that the eigenvalues are $\lambda = 1$ and $\lambda = 4$.

3. When $\lambda = 1$, we have

$$A-\lambda I = \begin{bmatrix}2-1 & 1 & 1\\1 & 2-1 & 1\\1 & 1 & 2-1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} \; r_1 := r_2 - \widetilde{r_1, r_3} := r_3 - r_1 \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

The solution of $(A-I)x = 0$ is $x_1 + x_2 + x_3 = 0$ and $x_1 = -x_2 - x_3$ So

$$Null(A-I) = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}.$$

The basis for the eigenspace corresponding to eigenvalue 1 is $\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$

4. When $\lambda = 4$, we have

$$A - \lambda I = \begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$interchange\ 1st\ row\ and\ 2nd\ row = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$r_2 := r_2 + 2r_1, r_3 := r_3 - r_1 \ = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} r_2 := r_2/3, r_2 : r_2 + r_3 \ = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_2 := r_1 + 2r_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

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The solution of $(A-4I)x = 0$ is $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$. This implies that $x_1 = x_3$, $x_2 = x_3$ and x_3 is free. So $Null(A-I) = \{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$.

The basis for the eigenspace corresponding to eigenvalue 4 is $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$

So A is diagonalizable with $A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Also $e^A = Pe^D P^{-1} = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^{-1}$.

12. Let B be the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the characteristic equation of A.

Solution: $B - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$.

So $det(B - \lambda I) = (2-\lambda)^2(1-\lambda)$. The characteristic equation of A is $(2-\lambda)^2(1-\lambda) = 0$.

(b) Find the eigenvalues and a basis of eigenvectors for B.

Solving $(2-\lambda)^2(1-\lambda) = 0$, we know that the eigenvalues of B are $\lambda = 2$ and $\lambda = 1$.

When $\lambda = 2$, we have

$$B - \lambda I = \begin{bmatrix} 2-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$r_2 := r_2 + \widetilde{r_3, r_1} := r_1 + r_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of $(B - 2I)x = 0$ is $x_2 = 0$, $x_3 = 0$ and x_1 is free. So $Null(B - 2I) = \{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$.

The basis for the eigenspace corresponding to eigenvalue 2 is $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$.

When $\lambda = 1$, we have

$$B-\lambda I=\begin{bmatrix}2-1 & 1 & 1\\ 0 & 2-1 & 1\\ 0 & 0 & 1-1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}$$

$$r_1 :\widetilde{=} r_1-r_2=\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}.$$

The solution of $(B-I)x=0$ is $x_1=0$ and $x_2+x_3=0$ So $x_1=0$, $x_2=-x_3$ and x_3 is free. $Null(B-I)=\{\begin{bmatrix}0\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}0\\-1\\1\end{bmatrix}\}$.

The basis for the eigenspace corresponding to eigenvalue 1 is $\{\begin{bmatrix}0\\-1\\1\end{bmatrix}$.

So $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is an eigenvector corresponding to eigenvalue 2 and $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is an eigenvector corresponding to eigenvalue 1 (c) Find a polynomial $P(B)$ in B such that $P(B)=0$. Verify your answer.

From (a), we have $p(\lambda)=det(B-\lambda I)=(2-\lambda)^2(1-\lambda)=-(\lambda-2)^2(\lambda-1)$. Then $P(B)=-(B-2I)^2(B-I)=0$. We can verify this by computing $B-2I=\begin{bmatrix}0&1&1\\0&0&1\\0&0&-1\end{bmatrix}$,$B-I=\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}$ and

$$-(B-2I)^2(B-I)=\begin{bmatrix}0&1&1\\0&0&1\\0&0&-1\end{bmatrix}\begin{bmatrix}0&1&1\\0&0&1\\0&0&-1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}=\begin{bmatrix}0&1&1\\0&0&1\\0&0&-1\end{bmatrix}\begin{bmatrix}0&1&1\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

(d) Diagonalize the matrix B if possible.

From (b), we know that B has only two independent eigenvectors and B is not diagonzalizable.