Solutions to Linear Algebra Practice Problems 1

1. Determine which of the following augmented matrices are in row echelon from, row reduced echelon form or neither. Also determine which variables are free if it's in row echelon form or row reduced echelon form.

$\begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ is neither row echelon from nor row reduced echelon form.

$\begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ is in row echelon from. Free variables is x_3.

$\begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ is in row echelon from. Free variables is x_2.

$\begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ is in row reduced echelon from. Free variables is x_2.

2. Show that $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix} \sim (r_2 := r_2 + 2r_1) \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$

$\sim \quad (r_3 := r_3 + 3r_1) \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 2 & -6 & -2 & 18 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix} \sim \quad (r_4 := r_4 - 3r_1)$

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 2 & -6 & -2 & 18 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix} \sim (r_3 := r_3 - 2r_2) \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix}$$

$$\sim \quad (r_3 \leftrightarrow r_4, r_3 := r_3/5) \quad \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \quad (r_2 := r_2 + r_3)$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \ (r_1 := r_1 - 2r_2) \ \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Determine if the following systems are consistent and if so give all solutions in parametric vector form.
(a)

$$\begin{aligned} x_1 \quad -2x_2 &= 3 \\ 2x_1 \quad -7x_2 &= 0 \\ -5x_1 \quad +8x_2 &= 5 \end{aligned}$$

Solution: The augmented matrix is $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_2 := r_2 - 2r_1)$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_3 := r_3 + 5r_1) \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & 20 \end{bmatrix}$$

$$\sim \quad (r_2 := r_2/-3, r_3 := r_3/-2) \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -10 \end{bmatrix} \sim \quad (r_3 := r_3 -$$

$r_2)$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$. The last row implies that $0 = -12$ which is impossible. So this system is inconsistent.

(b)

$$\begin{array}{lllll} x_1 & +2x_2 & -3x_3 & +x_4 & = 1 \\ -x_1 & -2x_2 & +4x_3 & -x_4 & = 6 \\ -2x_1 & -4x_2 & +7x_3 & -x_4 & = 1 \end{array}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim (r_2 := r_2 + r_1)$

$\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim (r_3 := r_3 + 2r_1)$ $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$

$\sim (r_3 := r_3 - r_2)$ $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 - r_3)$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$

$\sim (r_1 := r_1 - r_3)$ $\begin{bmatrix} 1 & 2 & -3 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 + 3r_2)$ $\begin{bmatrix} 1 & 2 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$.

So x_2 is free. The solution is $x_1 = 26 - 2x_2$, $x_3 = 7$, $x_4 = -47$. Its parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26 - 2x_2 \\ x_2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 2 & 6 & 5 & 1 \\ 3 & 9 & 4 & 5 \end{bmatrix}$.

(a) Find all the solutions of the non-homogeneous system $Ax = b$, and write them in parametric form, where $b = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$. Solution:

Consider the augmented matrix $[A \;\; b] = \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix}$.

Now we perform row operations on the augmented matrix.

$$\begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix} r_2 := (-2)r_1 + r_2, r_3 := (-3)r_1 + r_3 \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 13 & -13 & 0 \\ 0 & 0 & 16 & -16 & 0 \end{bmatrix}$$

$$r_2 := \tfrac{1}{13}r_2, r_3 := \tfrac{1}{16}r_3, r_3 := r_3 - r_2 \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r_1 := 4r_2 + r_1 \begin{bmatrix} 1 & 3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the solution is

$$\begin{cases} x_1 + 3x_2 + 3x_4 = -1 \\ x_3 - x_4 = 0 \\ x_2 \text{ and } x_4 \text{ are free.} \end{cases} \tag{1}$$

So

$$\begin{cases} x_1 = -1 - 3x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = x_4 \\ x_4 \text{ is free.} \end{cases} \tag{2}$$

Thus the solution of $Ax = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 - 3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

where x_2 and x_4 are any numbers.

(b) Find all the solutions of the homogeneous system $Ax = 0$, and write them in parametric form.
Solution:From previous example, we know that the solution of $Ax = 0$ is of the form

$$x = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are any numbers.

(c) Are the columns of the matrix A linearly independent? Write down a linear relation between the columns of A if they are dependent.
Solution: Since $Ax = 0$ has nontrivial solution, we know that the columns of the matrix A are linearly dependent. The solution is $x = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Choosing $x_2 = 1$ and $x_3 = 0$, We have s $x = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. This implies that

$$-3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + 0 \cdot \begin{bmatrix} -4 \\ 5 \\ 4 \end{bmatrix} + 0 \cdot \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5. Let $S = Span\Big\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -4 \end{bmatrix} \Big\}$.

(a) Find all the vectors $u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that the u is in S. Write these u in parametric form. Justify your answer.
Solution: Note that $u \in S$ iff the following system is consistent.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & a \\ -2 & 1 & -3 & 1 & b \\ 3 & 1 & 2 & 1 & c \\ 1 & -2 & 3 & -3 & d \end{bmatrix} \sim (r_2 := r_2 + 2r_1) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2\,a \\ 3 & 1 & 2 & 1 & c \\ 1 & -2 & 3 & -3 & d \end{bmatrix}$$

$$\sim \ (r_3 := r_3 - 3r_1) \ \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2\,a \\ 0 & 1 & -1 & 1 & c - 3\,a \\ 1 & -2 & 3 & -3 & d \end{bmatrix}$$

$$\sim \ (r_4 := r_4 - 3r_1) \ \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2\,a \\ 0 & 1 & -1 & 1 & c - 3\,a \\ 0 & -2 & 2 & -3 & d - a \end{bmatrix} \sim \ (r_3 := r_3 - r_2)$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2\,a \\ 0 & 0 & 0 & 0 & c - 5\,a - b \\ 0 & -2 & 2 & -3 & d - a \end{bmatrix} \sim (r_4 := r_4 + 2r_2) \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2\,a \\ 0 & 0 & 0 & 0 & c - 5\,a - b \\ 0 & 0 & 0 & -1 & d + 3\,a + 2\,b \end{bmatrix}$$

$$\sim \ (r_4 \leftrightarrow r_3, r_3 := -r_3) \ \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & -1 & 1 & b + 2\,a \\ 0 & 0 & 0 & 1 & -d - 3\,a - 2\,b \\ 0 & 0 & 0 & 0 & c - 5\,a - b \end{bmatrix}.$$ This system is consistent if $c - 5\,a - b = 0$. So $c = 5a + b$ and $u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} =$

$$\begin{bmatrix} a \\ b \\ 5a + b \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Is $v = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ in S.

Solution: We have $a = -1$, $b = 3$ and $c = -2$. So $c - 5a - b = -2 + 5 - 3 = 0$ So $v \in S$.

(c) Is $w = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ in S.

Solution: We have $a = 1$, $b = 3$ and $c = -2$. So $c - 5a - b = -2 - 5 - 3 = -10 \neq 0$ So v is not in S.

6. (a.)

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{array}\right] \quad \widetilde{r_1 \leftrightarrow r_2} \begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{bmatrix}$$

$$\widetilde{r_1 := -r_1} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{bmatrix} \quad \widetilde{r_2 := r_2 - 2r_1} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 6 \end{bmatrix}$$

$$\widetilde{r_2 := -r_3, r_3 := r_2} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 & 1 \\ 0 & 0 & -1 & 2 & 6 \end{bmatrix} \quad \widetilde{r_3 := r_3 - 3r_2} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & -3 & 1 \\ 0 & 0 & -1 & 2 & 6 \end{bmatrix}$$

$$\widetilde{r_3 := -r_4, r_4 := r_3} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & 4 & -3 & 1 \end{bmatrix} \quad \widetilde{r_4 := r_4 - 4r_3} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & 0 & 5 & 25 \end{bmatrix}$$

$$\widetilde{r_4 := r_4/5} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{matrix} \widetilde{} \\ r_2 := r_2 - r_4 \\ r_3 := r_3 + 2r_4 \end{matrix} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\begin{matrix} \widetilde{} \\ r_1 := r_1 - r_3 \\ r_2 := r_2 + 2r_3 \end{matrix} \begin{bmatrix} 1 & -2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \quad \widetilde{r_1 := r_1 + 2r_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Thus $(x_1, x_2, x_3, x_4) = (2, 3, 4, 5)$.

(b) The solution in part (a) implies that

$$2\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}+3\begin{bmatrix}-1\\2\\-1\\0\end{bmatrix}+4\begin{bmatrix}0\\-1\\2\\-1\end{bmatrix}+5\begin{bmatrix}0\\0\\-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\\0\\6\end{bmatrix}.$$

(c) Are the columns of $\begin{bmatrix}2 & -1 & 0 & 0\\-1 & 2 & -1 & 0\\0 & -1 & 2 & -1\\0 & 0 & -1 & 2\end{bmatrix}$ linearly independent?

The computation in (a) implies that the homogeneous equation has only trivial solution $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$. In particular, the span of columns of the matrix is R^4

7. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\1 & 2 & 1 & b\\3 & 5 & a & 1\end{array}\right]$$

(a) For what values of a will the system have a unique solution? What is the solution?(your answer may involve a and b)

(b) For what values of a and b will the system have infinitely many solutions?

(c) For what values of a and b will the system be inconsistent?

Answer:

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\1 & 2 & 1 & b\\3 & 5 & a & 1\end{array}\right] \sim (r_2 := r_2 + (-1)r_1) \begin{bmatrix}1 & 1 & 1 & 2\\0 & 1 & 0 & b-2\\3 & 5 & a & 1\end{bmatrix}$$

$$\sim (r_3 := r_3 + (-3)r_1) \begin{bmatrix}1 & 1 & 1 & 2\\0 & 1 & 0 & b-2\\0 & 2 & a-3 & -5\end{bmatrix}$$

$$\sim (r_3 := r_3 + (-2)r_2) \begin{bmatrix}1 & 1 & 1 & 2\\0 & 1 & 0 & b-2\\0 & 0 & a-3 & -1-2b\end{bmatrix}.$$

(a) If $a-3 \neq 0$ then previous augmented matrix is row equivalent to ($r_3 := \frac{1}{a-3} r_3$) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & 1 & \frac{-1-2b}{a-3} \end{bmatrix} \sim$ ($r_1 := r_1 - r_3$)

$$\begin{bmatrix} 1 & 1 & 0 & 2+\frac{1+2b}{a-3} \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & 1 & \frac{-1-2b}{a-3} \end{bmatrix}$$

$$\sim \; (r_1 := r_1 - r_2) \begin{bmatrix} 1 & 0 & 0 & 4+\frac{1+2b}{a-3}-b \\ 0 & 1 & 0 & b-2 \\ 0 & 0 & 1 & \frac{-1-2b}{a-3} \end{bmatrix}$$

The system will have a unique solution when $a \neq 3$. The solution is $x_1 = 4 + \frac{1+2b}{a-3} - b$, $x_2 = b-2$ and $x_2 = \frac{-1-2b}{a-3}$.

(b) The system will have infinitely many solutions if $a-3=0$ and $-1-2b=0$, i.e $a=3$ and $b=-\frac{1}{2}$.

(c) The system will be inconsistent if $a-3=0$ and $-1-2b \neq 0$, i.e $a=3$ and $b \neq -\frac{1}{2}$.

8. (a) $\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 0 & 0 \\ -6 & 3 \end{bmatrix} = [v_1 \; v_2]$. We have $v_1 = -2v_2$. So the set of column vectors is linearly dependent.

(b) $\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 2 & 2 \end{bmatrix}$. The first column vector is not a multiple of the second column vector. So the set of column vectors is linearly independent.

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} r_3 := r_3 - \widetilde{r_1, r_4} := r_4 - r_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$$\sim r_2 := -r_4, r_4 := r_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim r_3 := r_3 + r_2, r_4 := r_4 - r_2$$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim r_4 := r_4 - r_3, r_1 := r_1 - r_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So $x_1 = x_2 = x_3 = 0$. Thus the columns of the matrix is linearly independent.

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

(d)

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \underset{\substack{r_4 := r_4 - r_1 \\ r_3 := r_3 - r_1}}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\underset{\substack{r_3 := r_3 + 2r_2 \\ r_4 := r_4 + r_2}}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 := r_1 - r_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has only two pivot vectors. We have one free variable. So the columns of the matrix form a linearly dependent set.

(e)

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in R^4. We will have at least one free variable for the solution of $Ax = 0$.

9. (a)

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a+1 & 3 \\ 1 & a & a+1 \end{bmatrix} \underbrace{r_2 := r_2 + (-1)r_1, r_3 := r_3 + (-1)r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & a & 1 \\ 0 & a-1 & a-1 \end{bmatrix}$$

$$\underbrace{r_3 := \frac{1}{a-1} r_3 \ if a-1 \neq 0, r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & a & 1 \end{bmatrix} \underbrace{r_3 := r_3 + (-a)r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-a \end{bmatrix}$$

Thus the column vectors are independent if $a \neq 1$.
(b) The column vectors are dependent if $a = 1$.

10. First, note that

$$T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + T(2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + T(3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$$
$$= T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + 2T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + 3T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T(e_1) + 2T(e_2) + 3T(e_3).$$

We need to find $T(e_1)$, $T(e_2)$ and $T(e_3)$.
Since T is linear, we have $T(e_1 + e_2) = T(e_1) + T(e_2)$, $T(e_1 - e_2) = T(e_1) - T(e_2)$ and $T(e_1+e_2+e_3) = T(e_1)+T(e_2)+T(e_3)$. The conditions $T(e_1 + e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T(e_1 - e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T(e_1 + e_2 + e_3) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ can be written as

$$\begin{cases} T(e_1) + T(e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ T(e_1) - T(e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ T(e_1) + T(e_2) + T(e_3) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases} \tag{3}$$

Adding $T(e_1)+T(e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $T(e_1)-T(e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we get $2T(e_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $T(e_1) = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$. Similarly, $T(e_1) + T(e_2) - (T(e_1) - T(e_2)) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So $2T(e_2) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$. From $T(e_1) + T(e_2) + T(e_3) - (T(e_1) + T(e_2)) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we get $T(e_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Hence $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = T(e_1)+2T(e_2)+3T(e_3) = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} + 2\cdot \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix} + 3\cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -6 \end{bmatrix}$.