Linear Algebra (Math 2890) Practice Problems 1

1. Determine which of the following augmented matrices are in row echelon from, row reduced echelon form or neither. Also determine which variables are free if it's in row echelon form or row reduced echelon form.
$$\begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

2. Show that $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3. Determine if the following systems are consistent and if so give all solutions in parametric vector form.
(a)
$$\begin{aligned} x_1 \quad -2x_2 &= 3 \\ 2x_1 \quad -7x_2 &= 0 \\ -5x_1 \quad +8x_2 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} x_1 \quad +2x_2 \quad -3x_3 \quad +x_4 &= 1 \\ -x_1 \quad -2x_2 \quad +4x_3 \quad -x_4 &= 6 \\ -2x_1 \quad -4x_2 \quad +7x_3 \quad -x_4 &= 1 \end{aligned}$$

4. Let $A = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 2 & 6 & 5 & 1 \\ 3 & 9 & 4 & 5 \end{bmatrix}$.

   (a) Find all the solutions of the non-homogeneous system $Ax = b$, and write them in parametric form, where $b = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$.

   (b) Find all the solutions of the homogeneous system $Ax = 0$, and write them in parametric form.

(c) Are the columns of the matrix A linearly independent? Write down a linear relation between the columns of A if they are dependent.

5. Let $S = Span\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -4 \end{bmatrix} \right\}.$

(a) Find all the vectors $u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that the u is in S. Write these u in parametric form. Justify your answer.

(b) Is $v = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ in S.

(c) Is $w = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$ in S.

6. (a) Solve the following system of equations:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

(b) Explain your answer in (a) in terms of the column vectors of the corresponding matrix.

(c) Are the columns of $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ linearly independent?

7. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ 3 & 5 & a & 1 \end{array}\right]$$

(a) For what values of a will the system have a unique solution? What is the solution?(your answer may involve a and b)

(b) For what values of a and b will the system have infinitely many solutions?

(c) For what values of a and b will the system be inconsistent?

8. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 0 & 0 \\ -6 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 4 & -2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.$$

9. $M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a+1 & 3 \\ 1 & a & a+1 \end{bmatrix}.$

(a) Describe the values of a so that the column vectors of M are linearly independent.

(b) Describe the values of a so that the column vectors of M are linearly dependent.

10. Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Suppose $T : R^3 \mapsto R^2$ is a linear transformation such that $T(e_1 + e_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T(e_1 - e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T(e_1 + e_2 + e_3) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. What is $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix})$?