Linear Algebra (Math 2890) Solution to Final Review Problems

1. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.

   (a) Find the condition on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $Ax = b$ is solvable.

   Solution:

   Consider the augmented matrix $[A \;\; b] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 1 & 1 & 0 & b_2 \\ 0 & 1 & 2 & b_3 \\ -1 & 0 & -1 & b_4 \end{array}\right]$

   $$a_2 := a_2 + (-1)a_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -1 & -2 & b_2 - b_1 \\ 0 & 1 & 2 & b_3 \\ -1 & 0 & -1 & b_4 \end{array}\right]$$

   $$a_4 := a_4 + a_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -1 & -2 & b_2 - b_1 \\ 0 & 1 & 2 & b_3 \\ 0 & 2 & 1 & b_4 + b_1 \end{array}\right]$$

   $$a_2 := -a_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 1 & 2 & b_3 \\ 0 & 2 & 1 & b_4 + b_1 \end{array}\right]$$

$$\overset{\textstyle a_3 := a_3 - a_2,\ a_4 := a_4 - 2a_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \\ 0 & 0 & -3 & b_4 - b_1 + 2\,b_2 \end{array}\right]$$

$$\overset{\textstyle a_3 \leftrightarrow a_4}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 0 & -3 & b_4 - b_1 + 2\,b_2 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{array}\right]$$

From here, we can see that $Ax = b$ has a solution if $b_3 + b_2 - b_1 = 0$.

(b) What is the column space of A?

Solution:

The column space is the subspace spanned by the column vectors. From the computation in (a), we know that the column vectors of A are independent. So $Col(A) = span\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}\}$.

(c) Describe the subspace $col(A)^\perp$ and find an basis for $col(A)^\perp$.

Solution: $col(A)^\perp = \{x | x \cdot y = 0 \ for\ all\ y \in col(A)\}$

$$= \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 0\}$$

$$= \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | x_1 + x_2 - x_4 = 0, 2x_1 + x_2 + x_3 = 0, 2x_1 + 2x_3 - x_4 = 0\}$$

Consider $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \overset{r_2 := r_2 - 2r_1}{\sim} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{bmatrix}$

$$\overset{\sim}{r_3 := r_3 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & 2 & 1 \end{bmatrix} \overset{\sim}{r_2 := -r_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 2 & 1 \end{bmatrix}$$

$$\overset{\sim}{r_3 := r_3 + 2r_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \overset{\sim}{r_1 := r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\overset{\sim}{r_3 := r_3/(-3)} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{\sim}{r_1 := r_1 - r_3, r_2 := r_2 + 2r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So $x_1 + x_3 = 0$, $x_2 - x_3 = 0$ and $x_4 = 0$, x_3 is free. This implies that $x_1 = -x_3$, $x_2 = x_3$, $x_4 = 0$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Hence $col(A)^{\perp} = span\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$ and $\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$ is a basis for $col(A)^{\perp}$.

The dimension of $col(A)^{\perp}$ is 1.

(d) Use Gram-Schmidt process to find an orthogonal basis for the column of the matrix A.

Solution:

Let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $w_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}$.

Gram-Schmidt process is

$v_1 = w_1$, $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$ and $v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2$.

So $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Compute $w_2 \cdot v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$, $v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$ and $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Compute $w_3 \cdot v_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$, $w_3 \cdot v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3$,

$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3$ and

$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2-1-1 \\ 0-1-0 \\ 2-0-1 \\ -1+1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Hence $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \}$ is an orthogonal basis for $Col(A)$.

(e) Find an orthonormal basis for the column of the matrix A.
Solution:
Note that $||v_1|| = \sqrt{v_1 \cdot v_1} = \sqrt{3}$, $||v_2|| = \sqrt{v_2 \cdot v_2} = \sqrt{3}$ and $||v_3|| = \sqrt{v_3 \cdot v_3} = \sqrt{3}$. Hence $\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||} \} = \{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \}$
is an orthonormal basis for $Col(A)$.

(f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of A and write $y = \widehat{y} + z$ where $\widehat{y} \in col(A)$ and $z \in col(A)^{\perp}$. Also find the shortest distance from y to $Col(A)$.

Solution: Since $\{v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}\}$ is an orthogonal basis for $Col(A)$, $y = \widehat{y} + z$ where $\widehat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \in Col(A)$ and $z = y - \widehat{y} \in Col(A)^{\perp}$. Compute $y \cdot v_1 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 7+3+0+2 = 12$, $v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 1 + 1 = 3$, $y \cdot v_2 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 7 + 0 + 10 - 2 = 15$,

$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3,$

$y \cdot v_3 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0-3+10+2 = 9$, $v_3 \cdot v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3.$

So $\widehat{y} = \frac{12}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{(15)}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{9}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4+5+0 \\ 4+0-3 \\ 0+5+3 \\ -4+5-3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix}$ and

$z = y - \widehat{y} = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} - \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. Note that $z \in Col(A)^{\perp} = span\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\}$.

The shortest distance from y to $Col(A) = ||y - \widehat{y}|| = ||z|| = \sqrt{(2)^2 + (-2)^2 + (2)^2 + (0)^2} = \sqrt{12}$.

(g) Using previous result to explain why $Ax = y$ has no solution.
Solution: Since the orthogonal projection of y to $Col(A)$ is not y, this implies that y is not in $Col(A)$. So $Ax = y$ has no solution.

(h) Use orthogonal projection to find the least square solution of $Ax = y$.
Solution: The least square solution of $Ax = y$ is the solution of $Ax = \widehat{y} = \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix}$ where $\widehat{y}$ is the orthogonal projection of y onto the column space of A (from part (f), we know $\widehat{y} = \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix}$.)
Consider the augmented matrix

$$[A\,\widehat{y}] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 8 \\ -1 & 0 & -1 & -2 \end{array}\right] \overbrace{r_2 := r_2 - r_1, r_3 := r_3 + r_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 1 & 7 \end{array}\right]$$

$$\overbrace{r_3 := r_3 + r_2, r_4 := r_4 + r_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$$r_2 := -r_2, r_4 := \widetilde{r_4/(-3)}, r_3 \leftrightarrow r_4 \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_2 := r_2 - \widetilde{2r_3, r_1} := r_1 - 2r_3 \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_1 \widetilde{:= r_1} - 2r_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_1 = -1$, $x_2 = 2$, $x_3 = 3$ and the least square solution of $Ax = y$ is $x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(i) Use normal equation to find the least square solution of $Ax = y$.
Solution: The normal equation is $A^T Ax = A^T y$. Compute $A^T A =$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 6 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

and $A^T y = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ 27 \\ 36 \end{bmatrix}$.

So the normal equation $A^T Ax = A^T y$ is

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 6 & 6 \\ 3 & 6 & 9 \end{bmatrix} x = \begin{bmatrix} 12 \\ 27 \\ 36 \end{bmatrix}.$$

Consider the augmented matrix $\left[\begin{array}{ccc|c} 3 & 3 & 3 & 12 \\ 3 & 6 & 6 & 27 \\ 3 & 6 & 9 & 36 \end{array}\right] \sim$

$r_2 := r_2 - r_1, r_3 := r_3 - r_1$ $\left[\begin{array}{ccc|c} 3 & 3 & 3 & 12 \\ 0 & 3 & 3 & 15 \\ 0 & 3 & 6 & 24 \end{array}\right]$

$\sim r_3 := r_3 - r_2$ $\left[\begin{array}{ccc|c} 3 & 3 & 3 & 12 \\ 0 & 3 & 3 & 15 \\ 0 & 0 & 3 & 9 \end{array}\right] \sim r_1 := r_1/3, r_2 := r_2/3, r_3 :=$

$r_3/3$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\sim r_2 := r_2 - r_3,\ r_1 := r_1 - r_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\sim r_1 := r_1 - r_2,$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

So $x_1 = -1$, $x_2 = 2$, $x_3 = 3$ and the least square solution of $Ax = y$ is $x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

2. Find the equation $y = a + mx$ of the least square line that best fits the given data points. $(0, 1)$, $(1, 1)$, $(3, 2)$.

   Solution: We try to solve the equations $1 = a$, $1 = a + m$, $2 = a + 3m$, that is,
   $a = 1$, $a + m = 1$ and $a + 3m = 2$. It corresponding to the linear system

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$. We solve the normal equation

$$A^T A \begin{bmatrix} a \\ m \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Compute $A^T A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}$ and

$$A^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

Consider the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 4 & 3 \\ 4 & 11 & 8 \end{array}\right] \sim r_2 := r_2 - 2r_1 \left[\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & 3 & 2 \end{array}\right]$$

$$\sim r_2 := r_2/3 \left[\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & 1 & 2/3 \end{array}\right] \sim r_1 := r_1 - 4r_2 \left[\begin{array}{cc|c} 2 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{array}\right]$$

$$\sim r_1 := r_1/2 \left[\begin{array}{cc|c} 1 & 0 & 1/6 \\ 0 & 1 & 2/3 \end{array}\right]$$

So the least square solution is $a = 1/6$ and $m = 2/3$. The equation $y = 1/6 + 2/3x$ is the least square line that best fits the given data points. $(0,1)$, $(1,1)$, $(3,2)$.

3. Problem 3 will not be in the final exam. We don't have time to cover this topics.

4. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

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(a) Prove that $det(A-\lambda I)=(1-\lambda)^2(4-\lambda)$.

Solution: Compute $A-\lambda I=\begin{bmatrix}2-\lambda & 1 & 1\\ 1 & 2-\lambda & 1\\ 1 & 1 & 2-\lambda\end{bmatrix}$ and $det(A-\lambda I)=(2-\lambda)^3+1+1-(2-\lambda)-(2-\lambda)-(2-\lambda)=8-12\,\lambda+6\,\lambda^2-\lambda^3+2-6+3\lambda=-\lambda^3+6\lambda^2-9\lambda+4=(1-\lambda)^2(4-\lambda)$.

(b) Orthogonally diagonalizes the matrix A, giving an orthogonal matrix P and a diagonal matrix D such that $A=PDP^t$

Solution: We know that the eigenvalues are 1 ,1 and 4.

When $\lambda=1$, $A-(1)I=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$x\in Null(A-I)$ if $x_1+x_2+x_3=0$. So $x_1=-x_2-x_3$ and $x=\begin{bmatrix}-x_2-x_3\\ x_2\\ x_3\end{bmatrix}=x_2\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$. Thus $\{w_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}, w_2=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\}$ is a basis of eigenvectors when $\lambda=-1$.

Now we use Gram-Schmidt process to find an orthogonal basis for $Null(A-I)$.

Let $v_1=w_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$ and $v_2=w_2-\frac{w_2\cdot v_1}{v_1\cdot v_1}v_1$. Compute $w_2\cdot v_1=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\cdot\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}=1$ and $v_1\cdot v_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}\cdot\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}=2$.

So $v_2=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}-(\frac{1}{2})\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{2}\\ 1\end{bmatrix}$.

Hence $\{v_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}, v_2=\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{2}\\ 1\end{bmatrix}\}$ is an orthogonal basis for $Null(A-I)$.

When $\lambda = 4$, $A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\overbrace{interchange\ r_1\ and\ r_2}$,

$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\overbrace{-2\ r_1 + r_2, -r_1 + r_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$

$\overbrace{r_2 + r_3, r_2/(-3)} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overbrace{2r_2 + r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x \in Null(A-4I)$ if $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$. So $x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $\{v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ is a basis for $Null(A - 4I)$.

So $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ is an orthogonal basis for R^3 which are eigenvectors corresponding to $\lambda = 1$, $\lambda = 1$ and $\lambda = 4$. Compute $||v_1|| = \sqrt{2}$, $||v_2|| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$ and $||v_3|| = \sqrt{3}$.

Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \frac{v_3}{||v_3||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\}$ is an orthonormal basis for R^3 which are eigenvectors corresponding to $\lambda = 1$, $\lambda = 1$ and $\lambda = 4$.

Finally, we have $A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} P^T$ where $P = [\frac{v_1}{||v_1||}\ \frac{v_2}{||v_2||}\ \frac{v_3}{||v_3||}] =$

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(c) Find A^{10} and e^A.

So $A^{10} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{10} \end{bmatrix} P^T$ and $e^A = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^T$

5. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

(a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.

Let $Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2 = x^T \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} x$ and $A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$. We want to orthogonally diagonalizes A.

Compute $A - \lambda I = \begin{bmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (9-\lambda)(3-\lambda) - 16 = \lambda^2 - 12\lambda + 27 - 16 = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11)$. So $\lambda = 1$ or $\lambda = 11$. Since the eigenvalues of A are all positive, we know that the quadratic form is positive definite.

Now we diagonalize A.

$\lambda = 1$: $A - 1 \cdot I = \begin{bmatrix} 9-1 & -4 \\ -4 & 3-1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 1 \cdot I)$ iff $2x_1 - x_2 = 0$. So $x_2 = 2x_1$ and $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$.

$\lambda = 11$: $A - 11 \cdot I = \begin{bmatrix} 9-11 & -4 \\ -4 & 3-11 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 11 \cdot I)$ iff $x_1 + 2x_2 = 0$. So $x_1 = -2x_2$ and $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 11$.

Now $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1|| = \sqrt{5}$ and $||v_2|| = \sqrt{5}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T$ where $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Now $Q(x) = x^T A x = x^T P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T x = y^T \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} y = y_1^2 + 11y_2^2$ if $y = P^T x$. So $Py = PP^T x$, $x = Py$ and $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Note that we have used the fact that $PP^T = I$.

(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.

Let $Q(x_1, x_2) = -5x_1^2 + 4x_1x_2 - 2x_2^2 = x^T \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} x$ and $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. We want to orthogonally diagonalizes A.

Compute $A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (-5 - \lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+1)(\lambda+6)$. So $\lambda = -1$ or $\lambda = -6$. Since the eigenvalues of A are all negative, we know that the quadratic form is negative definite.

Now we diagonalize A.

$\lambda = -1$: $A - (-1)\cdot I = \begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$.

So $x \in Null(A - 1 \cdot I)$ iff $2x_1 - x_2 = 0$. So $x_2 = 2x_1$ and $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$.

$\lambda = -6$: $A - (-6)\cdot I = \begin{bmatrix} -5-(-6) & 2 \\ 2 & (-2)-(-6) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

So $x \in Null(A - 11 \cdot I)$ iff $x_1 + 2x_2 = 0$. So $x_1 = -2x_2$ and $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -6$.

Now $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1|| = \sqrt{5}$ and $||v_2|| = \sqrt{5}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = P \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} P^T$

where $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Now $Q(x) = x^T A x = x^T P \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} P^T x = y^T \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} y = -y_1^2 - 6y_2^2$ if $y = P^T x$. So $Py = PP^T x$, $x = Py$ and $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

(c) $8x_1^2 + 6x_1x_2$.

Let $Q(x_1, x_2) = 8x_1^2 + 6x_1x_2 = x^T \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix} x$ and $A = \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}$. We want to orthogonally diagonalizes A.

Compute $A - \lambda I = \begin{bmatrix} 8-\lambda & 3 \\ 3 & 0-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (8-\lambda) \cdot (-\lambda) - 9 = \lambda^2 - 8\lambda - 9 = (\lambda+1)(\lambda-9)$. So $\lambda = -1$ or $\lambda = 9$. Since A has positive and negative eigenvalues, we know that the quadratic form is indefinite.

Now we diagonalize A.

$\lambda = -1$: $A - (-1) \cdot I = \begin{bmatrix} 8-(-1) & 3 \\ 3 & 0-(-1) \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 1 \cdot I)$ iff $3x_1 + x_2 = 0$. So $x_2 = -3x_1$ and $x = \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$.

$\lambda = 9$: $A - 9 \cdot I = \begin{bmatrix} 8-9 & 3 \\ 3 & 0-9 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 9 \cdot I)$ iff $x_1 - 3x_2 = 0$. So $x_1 = 3x_2$ and $x = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 9$.

Now $\{v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1|| = \sqrt{10}$ and $||v_2|| = \sqrt{10}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{bmatrix}, \frac{v_2}{||v_2||} =$

$\begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = P\begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^T$ where $P = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$.

Now $Q(x) = x^TAx = x^TP\begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^Tx = y^T\begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} y = -y_1^2 + 9y_2^2$ if $y = P^Tx$. So $Py = PP^Tx$, $x = Py$ and $P\begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$.

6. (a) Show that the set of vectors

$$B = \left\{ u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right),\ u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right),\ u_3 = (0, 0, 1) \right\}$$

is an **orthonormal basis** of $\mathbb{R}^3$.

Solution: Compute $u_1 \cdot u_2 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{-12}{5} + \frac{12}{5} = 0$, $u_1 \cdot u_3 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot (0, 0, 1) = 0$, $u_2 \cdot u_3 = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot (0, 0, 1) = 0$, $u_1 \cdot u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right) = \frac{9}{25} + \frac{16}{25} = 1$, $u_3 \cdot u_3 = (0, 0, 1) \cdot (0, 0, 1) = 1$, $u_2 \cdot u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{16}{25} + \frac{9}{25} = 1$

(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis in (a).

Solution: Let $y = (1, -1, 2)$. So $y = \frac{y \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2}u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3}u_3 = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + (y \cdot u_3)u_3$. Compute $y \cdot u_1 = (1, -1, 2) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right) = -\frac{3}{5} - \frac{4}{5} = -\frac{7}{5}$, $y \cdot u_2 = (1, -1, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$, $y \cdot u_3 = (1, -1, 2) \cdot (0, 0, 1) = 2$.

So the coordinate of y with respect to the basis in (a) is $(-\frac{7}{5}, \frac{1}{5}, 2)$.

7. (a) Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Find the inverse matrix of A if possible.

Solution: Consider the augmented matrix $[A\, I] = \left[\begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$$\underset{r_1 := r_1 - r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{r_3 := r_3 - 2r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array}\right]$$

$$\underset{r_2 := r_2 + r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & -1 & -1 & -2 & 1 & 3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array}\right] \underset{r_2 := -r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array}\right]$$

$$\underset{r_3 := r_3 + 3r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right]$$

$$\underset{r_2 := r_2 - r_3,\, r_1 := r_1 - 3r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -11 & 9 & 17 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right]$$

$$\underset{r_1 := r_1 - 3r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$.

(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis B obtained from the column vectors of A.

Solution: The coordinate is $x = A^{-1}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} =$
$\begin{bmatrix} 8 \\ 2 \\ -5 \end{bmatrix}$.

8. Let A be an 3×3 matrix. Suppose $A^3 + 2A^2 - 4A + I = 0$. Is A invertible? Express A^{-1} in terms of A if possible.

   Solution: From $A^3 + 2A^2 - 4A + I = 0$, we have $A^3 + 2A^2 - 4A = -I$ and $A(A^2 + 2A - 4I) = -I$. Thus $A \cdot (-A^2 - 2A + 4I) = I$ and $A^{-1} = -A^2 - 2A + 4I$.

9. Find a basis for the subspace spanned by the following vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \right\}$.

   What is the dimension of the subspace?

   Solution: Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

   $r_2 := r_2 - \widetilde{r_1, r_3} := r_3 - r_1 \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -3 \\ 0 & -1 & -1 \end{bmatrix}$

   $r_2 \widetilde{:= r_2}/(-3) \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} r_3 \widetilde{:= r_3} + r_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   So the first two vectors are pivot vectors and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis.

   The dimension of the subspace is 2.

10. Determine if the following systems are consistent and if so give all solutions in parametric vector form.

(a)

$$\begin{aligned} x_1 \quad -2x_2 &= 3 \\ 2x_1 \quad -7x_2 &= 0 \\ -5x_1 \quad +8x_2 &= 5 \end{aligned}$$

Solution: The augmented matrix is $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_2 := r_2 - 2r_1)$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_3 := r_3 + 5r_1)$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & 20 \end{bmatrix}$ $\sim (r_2 := r_2/-3, r_3 := r_3/-2)$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -10 \end{bmatrix} \sim (r_3 := r_3 - r_2)$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$. The last row implies that $0 = -12$ which is impossible. So this system is inconsistent.

(b)

$$\begin{aligned} x_1 \quad +2x_2 \quad -3x_3 \quad +x_4 &= 1 \\ -x_1 \quad -2x_2 \quad +4x_3 \quad -x_4 &= 6 \\ -2x_1 \quad -4x_2 \quad +7x_3 \quad -x_4 &= 1 \end{aligned}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim (r_2 := r_2 + r_1)$ $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim (r_3 := r_3 + 2r_1)$ $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$

$$\sim (r_3 := r_3 - r_2) \begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 - r_3) \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$$

$$\sim (r_1 := r_1 - r_3) \begin{bmatrix} 1 & 2 & -3 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 + 3r_2) \begin{bmatrix} 1 & 2 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

So x_2 is free. The solution is $x_1 = 26 - 2x_2$, $x_3 = 7$, $x_4 = -47$. Its parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26 - 2x_2 \\ x_2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

11. Let $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$.

(a) Find a basis for the column space of A

(b) Find a basis for the nullspace of A

(c) Find the rank of the matrix A

(d) Find the dimension of the nullspace of A.

(e) Is $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ in the range of A?

(e) Does $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ have any solution? Find a solution if it's solvable.

Solution: Consider the augmented matrix $\left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 2 & -6 & 9 & -1 & 8 & 4 & 3 \\ 2 & -6 & 9 & -1 & 9 & 3 & 2 \\ -1 & 3 & -4 & 2 & -5 & 1 & 0 \end{array}\right]$

$$-2r_1 + r_2, \widetilde{-2r_1} + r_3, r_1 + r_4 \left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\widetilde{-r_2 + r_3} \left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$2r_3 + \widetilde{r_2, -5}r_3 + r_1 \left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 0 & 6 & 5 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\widetilde{-4r_2 +} r_1 \left[\begin{array}{ccccc|c|c} 1 & -3 & 0 & -14 & 0 & 6 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right].$$

So the first, third and fifth vector forms a basis for Col(A), i.e $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$ is a basis for Col(A). The rank of A is 3 and the dimension of the null space is $5 - 3 = 2$.

$x \in Null(A)$ if $x_1 - 3x_2 - 14x_4 = 0$, $x_3 + 3x_4 = 0$ and $x_5 = 0$. So $x = \begin{bmatrix} 3x_2 + 14x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Thus$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right.$ is a basis for $NULL(A)$.

From the result of row reduction, we can see that $Ax = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ is inconsistent (not solvable) and $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ is not in the range of A.

From the result of row reduction, we can see that $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ is solvable.

12. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.$$

Solution: $det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 - 1 = 1 \neq 0$. So the columns of the matrix form a linearly independent set.

$\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}$. The second column vector is a multiple of the first column vector. So the columns of the matrix form a linearly dependent set.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \quad \widetilde{interchange\ first\ and\ third\ row} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\widetilde{r_3 + 4r_1, r_4 + (-5)r_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \quad \widetilde{(-1)r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix}$$

$$\widetilde{r_3 + 3r_2, r_4 + (-4)r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \quad \widetilde{interchange\ 3rd\ and\ 4th\ row, \frac{1}{7}r_4} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in R^4.