Linear Algebra (Math 2890) Review Problems for Final Exam

1. Let \( A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \).

   (a) Find the condition on \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) such that \( Ax = b \) is consistent.

   (b) What is the column space of \( A \)?

   (c) Describe the subspace \( col(A)^\perp \) and find an basis for \( col(A)^\perp \). What’s the dimension of \( col(A)^\perp \)?

   (d) Use Gram-Schmidt process to find an orthogonal basis for the column space of \( A \).

   (e) Find an orthonormal basis for the column of the matrix \( A \).

   (f) Find the orthogonal projection of \( y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \) onto the column space of \( A \) and write \( y = \hat{y} + z \) where \( \hat{y} \in col(A) \) and \( z \in col(A)^\perp \). Also find the shortest distance from \( y \) to \( Col(A) \).

   (g) Using previous result to explain why \( Ax = y \) has no solution.

   (h) Use orthogonal projection to find the least square solution of \( Ax = y \).

   (i) Use normal equation to find the least square solution of \( Ax = y \).

2. Find the equation \( y = a + mx \) of the least square line that best fits the given data points. (0, 1), (1, 1), (3, 2).
3. (a) Find the singular values of \( A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \).

(b) Find the singular value decomposition of \( A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \).

4. Let \( A \) be the matrix
\[
A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.
\]

(a) Prove that \( \det(A - \lambda) = (1 - \lambda)^2(4 - \lambda) \).

(b) Orthogonally diagonalizes the matrix \( A \), giving an orthogonal matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^t \).

(c) Find \( A^{10} \) and \( e^A \).

5. Classify the quadratic forms for the following quadratic forms. Make a change of variable \( x = Py \), that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

(a) \( 9x_1^2 - 8x_1x_2 + 3x_2^2 \).

(b) \( -5x_1^2 + 4x_1x_2 - 2x_2^2 \).

(c) \( 8x_1^2 + 6x_1x_2 \).

6. (a) Show that the set of vectors
\[
B = \left\{ u_1 = \left( \frac{-3}{5}, \frac{4}{5}, 0 \right), \; u_2 = \left( \frac{4}{5}, \frac{3}{5}, 0 \right), \; u_3 = (0, 0, 1) \right\}
\]
is an orthonormal basis of \( \mathbb{R}^3 \).

(b) Find the coordinates of the vector \((1, -1, 2)\) with respect to the basis in (a).
7. (a) Let \( A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \). Find the inverse matrix of \( A \) if possible.

(b) Find the coordinates of the vector \((1, -1, 2)\) with respect to the basis \( B \) obtained from the column vectors of \( A \).

8. Let \( A \) be an \( 3 \times 3 \) matrix. Suppose \( A^3 + 2A^2 - 4A + I = 0 \). Is \( A \) invertible? Express \( A^{-1} \) in terms of \( A \) if possible.

9. Find a basis for the subspace spanned by the following vectors \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \} \).

What is the dimension of the subspace?

10. Determine if the following systems are consistent and if so give all solutions in parametric vector form.

(a)
\[
\begin{align*}
    x_1 & -2x_2 = 3 \\
    2x_1 & -7x_2 = 0 \\
    -5x_1 & +8x_2 = 5
\end{align*}
\]

(b)
\[
\begin{align*}
    x_1 & +2x_2 -3x_3 +x_4 = 1 \\
    -x_1 & -2x_2 +4x_3 -x_4 = 6 \\
    -2x_1 & -4x_2 +7x_3 -x_4 = 1
\end{align*}
\]

11. Let \( A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix} \).

(a) Find a basis for the column space of \( A \)
(b) Find a basis for the nullspace of \( A \)
(c) Find the rank of the matrix \( A \)
(d) Find the dimension of the nullspace of $A$.

(e) Is $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ in the range of $A$?

(e) Does $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ have any solution? Find a solution if it’s solvable.

12. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}$, $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$, $\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$. 