1. (a) What is a subspace in $R^n$?

Solution: A subspace of $R^n$ is any set $H$ in $R^n$ that satisfies the following three properties. (I) The zero vector is in $H$. (II) For each $u$ and $v$ in $H$, then $u + v$ is in $H$. (III) For each $u$ in $H$ and each scalar $c$, the vector $cu$ is in $H$.

(b) Is the set $\{(x, y, z)|x + y + z = 1\}$ a subspace?
Solution: This is not a subspace since the zero vector $(0, 0, 0)$ is not in the set.

(c) Is the set $\{(x, y, z)|x - y - z = 0, x + y - z = 0\}$ a subspace?
Solution: Yes. This is a subspace. This can be regarded as the nullspace of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

Here $Nul(A) = \{(x, y, z)| \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.

(d) What is a basis for a subspace?
Solution: A basis for a subspace $H$ of $R^n$ is a linearly independent set in $H$ that spans $H$.

(e) What is the dimension of a subspace?
Solution: The dimension of a nonzero subspace $H$ is the number of vectors in any basis for $H$.

(f) What is the column space of a matrix?
Solution: The column space of a matrix $A$ is the set of the span of the column vectors of $A$.

(g) What is the null space of a matrix?
Solution: The null space of a matrix $A$ is the set of all solutions to the homogeneous equation $Ax = 0$, i.e. $Nul(A) = \{x | Ax = 0\}$.

(h) What is an eigenvalue of a matrix $A$?
Solution: Let $A$ be a $n \times n$ matrix. A scalar $\lambda$ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.
(i) What is an eigenvector of a matrix $A$?
Solution: Let $A$ be a $n \times n$ matrix. A scalar $\lambda$ such that $Ax = \lambda x$ for some $x \neq 0$ is called an eigenvalue and the corresponding vector is called an eigenvector.

(j) What is the characteristic polynomial of a matrix $A$?
Solution: The polynomial $\det(A - \lambda I)$ is the characteristic polynomial of a matrix $A$.

(k) What is the subspace spanned by the vectors $v_1, v_2, \ldots, v_p$?
Solution: The subspace spanned by $v_1, v_2, \ldots, v_p$ is the set of all possible linear combination of $v_1, v_2, \ldots, v_p$, i.e. $\text{Span}\{v_1, \ldots, v_n\} = \{c_1v_1 + c_2v_2 + \cdots + c_pv_p | c_1, c_2, \ldots, c_p \text{ are real numbers}\}$

2. Find the inverses of the following matrices if they exist.

$A = \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -1 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$.

Solution: (a) Since $\det(A) = -1$, we have $A^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & -7 \end{bmatrix}$

(b)
\[ r_1 := r_1 + r_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ -1 & 0 & -1 \\ -3 & -1 & -5 \end{bmatrix} \]

So \( B^{-1} = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 0 & -1 \\ -3 & -1 & -5 \end{bmatrix} \).

(c)

\[ \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} r_2 := r_2 - 2r_1 \]

\[ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 8 & 9 & 10 \end{bmatrix} \]

\[ r_2 \leftrightarrow r_1 \]

\[ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 8 & 9 & 10 \end{bmatrix} \]

\[ r_2 := r_2 - 2r_1, r_3 := r_3 - 8r_1 \]

\[ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 6 \\ 0 & 9 & 18 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -5 & 1 \end{bmatrix} \]

So \( C \) only has one free variable (or two pivot vectors) and \( C \) is not invertible.

3. (a) Let \( A \) be an \( 3 \times 3 \) matrix. Suppose \( A^3 + 2A^2 - 3A + 4I = 0 \). Is \( A \) invertible? Express \( A^{-1} \) in terms of \( A \) if possible.

Solution: From \( A^3 + 2A^2 - 3A + 4I = 0 \), we have \( A^3 + 2A^2 - 3A = -4I \), \( A(A^2 + 2A - 3I) = -4I \) and \( A \cdot (-\frac{1}{4}(A^2 + 2A - 3I)) = I \). So \( A^{-1} = -\frac{1}{4}(A^2 + 2A - 3I) \).

(b) Suppose \( A^3 = 0 \). Is \( A \) invertible?

Solution: If \( A \) is invertible then \( A^{-2}A^3 = A^{-2}0 = 0 \) and \( A = 0 \) which is not invertible. So \( A \) is not invertible.

4. Find all values of \( a \) and \( b \) so that the subspace of \( \mathbb{R}^4 \) spanned by
\[
\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ -a \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \right\}
\] 

is two-dimensional.

Solution: Consider the matrix
\[
A = \begin{bmatrix}
0 & b & -2 \\
1 & 1 & 2 \\
0 & -a & 0 \\
-1 & 1 & 0
\end{bmatrix}
\]

Interchange first row and second row
\[
\begin{bmatrix}
1 & 1 & 2 \\
0 & b & -2 \\
0 & -a & 0 \\
-1 & 1 & 0
\end{bmatrix}
\]

\[
r_4 := r_1 + r_4
\]

Interchange second row and forth row
\[
\begin{bmatrix}
1 & 1 & 2 \\
0 & 2 & 2 \\
0 & -a & 0 \\
0 & b & -2
\end{bmatrix}
\]

Divide second row by 2
\[
\begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 1 \\
0 & -a & 0 \\
0 & b & -2
\end{bmatrix}
\]

\[
r_3 := r_3 + ar_2, r_4 := r_4 - br_2
\]

Now the first and second vectors are pivot vectors. So \( \text{rank}(A) = 2 \) if \( a = 0 \) and \( -2 - b = 0 \).

So \( a = 0 \) and \( b = -2 \)

5. Let \( \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \). You can assume that \( \mathcal{B} \) is a basis for \( \mathbb{R}^3 \)

(a) Which vector \( x \) has the coordinate vector \( [x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \).
Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. So $x = A[x]_B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - 3 + 0 \\ 0 - 2 + 0 \\ 0 - 1 + 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$

(b) Find the $\beta$-coordinate vector of $y = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

Solution. We have to solve $Ax = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \tilde{r}_2 := \frac{1}{2} r_2 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} r_2 := r_3 - r_2 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$

$\tilde{r}_3 := \frac{1}{2} r_3 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} r_1 := r_1 - 3 r_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

So $[y]_B = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$.

6. Let $M = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

(a) Find bases for $\text{Col}(M)$ and $\text{Nul}(M)$, and then state the dimensions of these subspaces.

Solution: $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} r_2 := -r_1 + r_2, r_3 := -r_2 + r_3 \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} r_3 := -2 r_2 + r_3 \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} r_1 := -2 r_2 + r_3 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So the first two vectors are pivot vectors and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis.
for Col(M) and dim(Col(M)) = 2.

The solution to \( Mx = 0 \) is \( x_1 + x_3 - x_4 = 0 \) and \( x_2 + 2x_3 + x_4 = 0 \). So

\[
x = \begin{bmatrix}
-x_3 + x_4 \\
-2x_3 - x_4 \\
x_3 \\
x_4
\end{bmatrix} = x_3 \begin{bmatrix}
-1 \\
-2 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
1 \\
1 \\
0 \\
1
\end{bmatrix}.
\]

Hence the basis for \( \text{Nul}(M) \) is \( \{ \begin{bmatrix}
-1 \\
-2 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
-1 \\
0 \\
1
\end{bmatrix} \} \) and \( \text{dim}(\text{Nul}(M)) = 2 \).

(b) Express the third column vector \( M \) as a linear combination of the basis of \( \text{Col}(M) \). From the row reduced echelon form, we know that \( \text{column}(3) = 1 \cdot \text{column}(1) + 2 \cdot \text{column}(2) \)

So

\[
\begin{bmatrix}
3 \\
5 \\
7
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} + 2 \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}.
\]

7. Find a basis for the subspace spanned by the following vectors \( \{ \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \begin{bmatrix}
3 \\
5 \\
7
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} \} \).

What is the dimension of the subspace?

Solution: Consider the matrix \( A = \begin{bmatrix}
1 & 1 & 3 & 0 \\
1 & 2 & 5 & 1 \\
1 & 3 & 7 & 2
\end{bmatrix} \)

From previous example, we know that the first two vectors are pivot vectors and \( \{ \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} \} \) is a basis. The dimension of the subspace is 2.

8. Determine which sets in the following are bases for \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Justify your answer

(a) \( \begin{bmatrix}
-1 \\
2
\end{bmatrix}, \begin{bmatrix}
2 \\
-4
\end{bmatrix} \).

Solution: Since \( \begin{bmatrix}
2 \\
-4
\end{bmatrix} = -2 \begin{bmatrix}
-1 \\
2
\end{bmatrix} \), the set \( \{ \begin{bmatrix}
-1 \\
2
\end{bmatrix}, \begin{bmatrix}
2 \\
-4
\end{bmatrix} \} \) is dependent. It is not a basis.
9. Let $A$ be the matrix

$$A = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

Find a polynomial $f(A)$ in $A$ such that $f(A) = 0$. Verify your answer.

Solution: 1. $A - \lambda I = \begin{bmatrix} -3 - \lambda & -4 \\ -4 & 3 - \lambda \end{bmatrix}$. 

So $\det(A - \lambda I) = (-3 - \lambda)(3 - \lambda) - 16 = \lambda^2 - 9 - 16 = \lambda^2 - 25$

2. Let $f(A) = A^2 - 25I$. Then $A^2 - 25I = 0$. One can verify this by checking $A^2 = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = 25 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 25I$. 

Hence $A^2 - 25I = 0$. 

10. Let $A$ be the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Prove that $\det(A - \lambda I) = -(\lambda - 1)^2(\lambda - 4)$.
(b) Find the eigenvalues and a basis of eigenvectors for $A$.
(c) Diagonalize the matrix $A$ if possible.
(d) Find an expression for $A^k$. 
(e) Find the matrix exponential $e^A$. Solution.
a. 1. \(A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 1 \\ 1 & 1 & 2 - \lambda \end{bmatrix}.\)

So \(\det(A - \lambda I) = (2 - \lambda)^3 + 1 + 1 - (2 - \lambda) - (2 - \lambda) - (2 - \lambda) = (4 - 4\lambda + \lambda^2)(2 - \lambda) + 2 - 6 + 3\lambda = 8 - 8\lambda + 2\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 4 + 3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda - 1)^2(\lambda - 4).\) So the characteristic equation is \(-(\lambda - 1)^2(\lambda - 4) = 0.\)

2. Solving the characteristic equation \(-(\lambda - 1)^2(\lambda - 4) = 0,\) we get that the eigenvalues are \(\lambda = 1\) and \(\lambda = 4.\)

3. When \(\lambda = 1,\) we have

\[
A - \lambda I = \begin{bmatrix}
2 - 1 & 1 & 1 \\
1 & 2 - 1 & 1 \\
1 & 1 & 2 - 1
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

so \(r_1 := r_2 - r_1, r_3 := r_3 - r_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.\)

The solution of \((A - I)x = 0\) is \(x_1 + x_2 + x_3 = 0\) and \(x_1 = -x_2 - x_3\) So

\[
\text{Null}(A - I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

The basis for the eigenspace corresponding to eigenvalue 1 is \(\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\)

4. When \(\lambda = 4,\) we have

\[
A - \lambda I = \begin{bmatrix} 2 - 4 & 1 & 1 \\ 1 & 2 - 4 & 1 \\ 1 & 1 & 2 - 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}
\]

interchange 1st row and 2nd row = \[
\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}
\]

\[
r_2 := r_2 + 2r_1, r_3 := r_3 - r_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} r_2 := r_2/3, r_2 : r_2 + r_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
r_2 := r_1 + 2r_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]
The solution of \((A - 4I)x = 0\) is \(x_1 - x_3 = 0\) and \(x_2 - x_3 = 0\). This implies that \(x_1 = x_3\), \(x_2 = x_3\) and \(x_3\) is free. So \(\text{Null}(A-I) = \{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \} \).

The basis for the eigenspace corresponding to eigenvalue 4 is \(\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \} \).

So \(A\) is diagonalizable with \(A = PDP^{-1}\) where \(P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\) and \(D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\).

Hence \(A^k = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^k \end{bmatrix} P^{-1}\).

Also \(e^A = P e^D P^{-1} = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^{-1}\).

11. Let \(B\) be the matrix \(\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\).

   (a) Find the characteristic equation of \(A\).

   Solution: \(B - \lambda I = \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 0 & 2 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{bmatrix}\).

   So \(det(B - \lambda I) = (2 - \lambda)^2(1 - \lambda)\). The characteristic equation of \(A\) is \((2 - \lambda)^2(1 - \lambda) = 0\).

   (b) Find the eigenvalues and a basis of eigenvectors for \(B\).
Solving \((2 - \lambda)^2(1 - \lambda) = 0\), we know that the eigenvalues of \(B\) are \(\lambda = 2\) and \(\lambda = 1\).

When \(\lambda = 2\), we have
\[
B - \lambda I = \begin{bmatrix} 2 - 2 & 1 & 1 \\ 0 & 2 - 2 & 1 \\ 0 & 0 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}
\]
\[r_2 := r_2 + r_3, r_1 := r_1 + r_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]
The solution of \((B - 2I)x = 0\) is \(x_2 = 0, x_3 = 0\) and \(x_1\) is free. So
\[
\text{Null}(B - 2I) = \{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}.
\]
The basis for the eigenspace corresponding to eigenvalue 2 is \(\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \).

When \(\lambda = 1\), we have
\[
B - \lambda I = \begin{bmatrix} 2 - 1 & 1 & 1 \\ 0 & 2 - 1 & 1 \\ 0 & 0 & 1 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
\[r_1 := r_1 - r_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]
The solution of \((B - I)x = 0\) is \(x_1 = 0\) and \(x_2 + x_3 = 0\) So \(x_1 = 0, x_2 = -x_3\) and \(x_3\) is free. \(\text{Null}(B - I) = \{ \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \}.
\]
The basis for the eigenspace corresponding to eigenvalue 1 is \(\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \} \).

So \(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\) is an eigenvector corresponding to eigenvalue 2 and \(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\) is an eigenvector corresponding to eigenvalue 1.

(c) Diagonalize the matrix B if possible.
From (b), we know that $B$ has only two independent eigenvectors and $B$ is not diagonalizable.

12. Find an basis for $W^\perp$ for the following $W$. Verify your answer.

(a) $W = \text{Span}\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \}$.

Solution: $W^\perp = \{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0 \}$

So $-x_1 + 2x_2 + x_3 = 0$ and $x_1 = 2x_2 + x_3$. Hence $x = \begin{bmatrix} 2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So $\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for $W^\perp$.

We can check that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -2 + 2 = 0$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 + 1 = 0$.

(b) $W = \text{Span}\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix} \}$.

Solution: $W^\perp = \{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0, x \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 0 \}$

So $-x_1 + 2x_2 + x_3 = 0$ and $2x_1 - 3x_2 + x_3 = 0$. Hence $W^\perp = \text{Null}(A)$ where $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -3 & 1 \end{bmatrix}$.

Perform row reduction on $\begin{bmatrix} -1 & 2 & 1 & 0 \\ 2 & -3 & 1 & 0 \end{bmatrix}$

$\sim (r_1 = -r_1, r_2 = r_2 + 2r_2) \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \sim (r_1 = r_1 + 2r_2) \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$

So $x_1 + 5x_3 = 0$ and $x_2 + 3x_3 = 0$. Hence $x_1 = -5x_3$ and $x_2 = -3x_3$. 
\[ x = \begin{bmatrix} -5x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} . \text{ So } \left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } W^\perp. \]

We can check that
\[ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 5 - 6 + 1 = 0 \text{ and } \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = -10 + 9 + 1 = 0. \]

13. (a) Let \( W = \text{Span}\{u_1, u_2\} \) where \( u_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \). Show that \( \{u_1, u_2\} \) is an orthogonal basis for \( W \).

Solution: Compute \( u_1 \cdot u_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = -2 + 4 - 2 = 0 \). So \( \{u_1, u_2\} \) is an orthogonal basis for \( W \).

(b) Find the closest point to \( y = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \) in the subspace \( W \).

Solution: The closest point to \( y = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \) in the subspace \( W \) is
\[
\text{Proj}_W(y) = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2.
\]

Compute \( y \cdot u_1 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 1 + 10 - 2 = 9, \]
\( u_1 \cdot u_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = 1 + 4 + 4 = 9, \]
\( y \cdot u_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -2 + 10 + 1 = 9, \]
\( u_2 \cdot u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 + 4 + 1 = 9. \]

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So \( \text{Proj}_W(y) = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{9}{9} u_1 + \frac{9}{5} u_2 = u_1 + u_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \).

(c) Find the distance between the point \( y \) and the subspace \( W \).

The distance between \( y \) and the subspace \( W \) is \( ||y - \text{Proj}_W(y)|| \). Compute \( y - \text{Proj}_W(y) = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \) and

\[ ||y - \text{Proj}_W(y)|| = \| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \| = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{9} = 3. \]

Hence the distance between the point \( y \) and the subspace \( W \) is 3.