Linear Algebra (Math 2890) Solution to Final Review Problems

1. Let $A$ be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$ 

(a) Prove that $\det(A - \lambda I) = (1 - \lambda)^2(4 - \lambda)$.

Solution: Compute $A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 1 \\ 1 & 1 & 2 - \lambda \end{bmatrix}$ and

$$\det(A - \lambda I) = (2 - \lambda)^3 + 1 + 1 - (2 - \lambda) - (2 - \lambda) - (2 - \lambda) = 8 - 12\lambda + 6\lambda^2 - \lambda^3 + 2 - 6 + 3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (1 - \lambda)^2(4 - \lambda).$$

(b) Orthogonally diagonalizes the matrix $A$, giving an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^t$.

Solution: We know that the eigenvalues are 1,1 and 4.

When $\lambda = 1$, $A - (1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} ~ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$x \in \text{Null}(A - I)$ if $x_1 + x_2 + x_3 = 0$. So $x_1 = -x_2 - x_3$ and

$x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Thus $\{w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis of eigenvectors when $\lambda = -1$.

Now we use Gram-Schmidt process to find an orthogonal basis for $\text{Null}(A - I)$.

Let $v_1 = w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $w_2 \cdot v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1$ and $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2$.

So $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{2}\right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.
Hence \( \{ v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \} \) is an orthogonal basis for \( \text{Null}(A-I) \).

When \( \lambda = 4 \), \( A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \)

interchange \( r_1 \) and \( r_2 \),

\[
\begin{bmatrix}
1 & -2 & 1 \\
-2 & 1 & 1 \\
1 & 1 & -2 \\
\end{bmatrix}
\]

\(-2 r_1 + r_2, -r_1 + r_3 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \)

\( r_2 + r_3, r_2/(-3) \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} 2r_2 + r_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} x \in \text{Null}(A-4I) \) if \( x_1 - x_3 = 0 \) and \( x_2 - x_3 = 0 \). So \( x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Thus \( \{ v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \} \) is a basis for \( \text{Null}(A-4I) \).

So \( \{ v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \} \) is an orthogonal basis for \( \mathbb{R}^3 \) which are eigenvectors corresponding to \( \lambda = 1 \), \( \lambda = 1 \) and \( \lambda = 4 \). Compute \( ||v_1|| = \sqrt{2}, ||v_2|| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}} \) and \( ||v_3|| = \sqrt{3} \).

Thus \( \{ \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{2}} \\ 0 \end{bmatrix}, \frac{v_3}{||v_3||} = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{3}} \end{bmatrix} \} \) is an orthonormal basis for \( \mathbb{R}^3 \) which are eigenvectors corresponding to
\[ \lambda = 1, \lambda = 1 \text{ and } \lambda = 4. \]

Finally, we have \( A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} P^T \) where \( P = \begin{bmatrix} v_1 & ||v_1|| & v_2 & ||v_2|| & v_3 & ||v_3|| \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}. \)

(c) Write the quadratic form associated with \( A \) using variables \( x_1, x_2, \) and \( x_3? \)

Solution: Recall that the quadratic form in \( x_1, x_2, \) and \( x_3 \) is \( Q_A(x) = x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3. \)

(d) Find \( A^{-1}, A^{10} \) and \( e^A. \)

Solution: Recall that \( A = PDP^T. \) Then \( A^{-1} = PD^{-1}P^T = P \begin{bmatrix} 1^{-1} & 0 & 0 \\ 0 & 1^{-1} & 0 \\ 0 & 0 & 4^{-1} \end{bmatrix} P^T = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} P^T. \)

\[ A^{10} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{10} \end{bmatrix} P^T \text{ and } e^A = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^T. \)

(e) What’s \( A^{-5}( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix})? \)

Solution: Note that \( v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) is an eigenvector with eigenvalue \( 4. \)

So we have \( A( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( A^k( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 4^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \) Hence \( A^{-5}( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 4^{-5} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \)

(f) What is \( \lim_{n \to \infty} A^{-n}? \) Recall that \( A = PDP^T \) and \( A^{-n} = PD^{-n}P^T = P \begin{bmatrix} 1^{-n} & 0 & 0 \\ 0 & 1^{-n} & 0 \\ 0 & 0 & 4^{-n} \end{bmatrix} P^T = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{-n} \end{bmatrix} P^T. \) Note that \( \lim_{n \to \infty} 4^{-n} = \ldots \)
0. So we have \( \lim_{n \to \infty} A^{-n} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T. \)

2. Classify the quadratic forms for the following quadratic forms. Make a change of variable \( x = Py \), that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

(a) \( 9x_1^2 - 8x_1x_2 + 3x_2^2. \)

Let \( Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2 = x^T \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} x \) and \( A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}. \) We want to orthogonally diagonalizes \( A \).

Compute \( A - \lambda I = \begin{bmatrix} 9 - \lambda & -4 \\ -4 & 3 - \lambda \end{bmatrix} \) and \( \text{det}(A - \lambda I) = (9 - \lambda)(3 - \lambda) - 16 = \lambda^2 - 12\lambda + 27 - 16 = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11). \) So \( \lambda = 1 \) or \( \lambda = 11. \) Since the eigenvalues of \( A \) are all positive, we know that the quadratic form is positive definite.

Now we diagonalize \( A. \)

\( \lambda = 1: \) \( A - 1 \cdot I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \) and \( \text{det}(A - 1 \cdot I) = 2 - 1 \cdot 2 = (2 - 1) \cdot 2 \).

So \( x \in \text{Null}(A - 1 \cdot I) \) iff \( 2x_1 - x_2 = 0. \) So \( x_2 = 2x_1 \) and \( x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} \).

\( \lambda = 11: \) \( A - 11 \cdot I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \) and \( \text{det}(A - 11 \cdot I) = 2 - 1 \cdot 2 \).

So \( x \in \text{Null}(A - 11 \cdot I) \) iff \( x_1 + 2x_2 = 0. \) So \( x_1 = -2x_2 \) and \( x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \).

So \( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \) is an eigenvector corresponding to eigenvalue \( \lambda = 11. \)

Now \( \{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\} \) is an orthogonal basis. Compute \( ||v_1|| = \sqrt{5} \) and \( ||v_2|| = \sqrt{5}. \) Thus \( \left\{ \frac{v_1}{||v_1||} = \begin{bmatrix} 1 \sqrt{5} \\ 2 \sqrt{5} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} -2 \sqrt{5} \\ 1 \sqrt{5} \end{bmatrix} \right\} \) is
an orthonormal basis of eigenvectors. So we have $A = P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T$

where $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{\sqrt{5}}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$.

Now $Q(x) = x^T Ax = x^T P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T x = y^T \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} y = y_1^2 + 11y_2^2$ if $y = P^T x$. So $Py = PP^T x$, $x = Py$ and $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{\sqrt{5}}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$.

Note that we have used the fact that $PP^T = I$.

(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.

Let $Q(x_1, x_2) = -5x_1^2 + 4x_1x_2 - 2x_2^2 = x^T \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} x$ and $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. We want to orthogonally diagonalize $A$.

Compute $A - \lambda I = \begin{bmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{bmatrix}$ and $\text{det}(A - \lambda I) = (-5 - \lambda)(-2 - \lambda) - 4 = \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = (\lambda + 1)(\lambda + 6)$.

So $\lambda = -1$ or $\lambda = -6$. Since the eigenvalues of $A$ are all negative, we know that the quadratic form is negative definite.

Now we diagonalize $A$.

$\lambda = -1$: $A - (-1)I = \begin{bmatrix} -5 - (-1) & 2 \\ 2 & -2 - (-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$.

So $x \in \text{Null}(A + 1 \cdot I)$ iff $2x_1 - x_2 = 0$. So $x_2 = 2x_1$ and $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$.

$\lambda = -6$: $A - (-6)I = \begin{bmatrix} -5 - (-6) & 2 \\ 2 & (-2) - (-6) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

So $x \in \text{Null}(A - 11 \cdot I)$ iff $x_1 + 2x_2 = 0$. So $x_1 = -2x_2$ and $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -6$.

Now $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute
\[ \|v_1\| = \sqrt{5} \text{ and } \|v_2\| = \sqrt{5}. \] Thus \( \frac{v_1}{\|v_1\|} = \left[ \frac{1}{\sqrt{5}} \right], \frac{v_2}{\|v_2\|} = \left[ \frac{-2}{\sqrt{5}} \right] \) is an orthonormal basis of eigenvectors. So we have \( A = P\left[ \begin{array}{cc} -1 & 0 \\ 0 & -6 \end{array} \right] P^T \)

where \( P = \left[ \begin{array}{cc} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{array} \right] \).

Now \( Q(x) = x^T Ax = x^T P\left[ \begin{array}{cc} -1 & 0 \\ 0 & -6 \end{array} \right] P^T x = y^T \left[ \begin{array}{cc} -1 & 0 \\ 0 & -6 \end{array} \right] y = -y_1^2 - 6y_2^2 \) if \( y = P^T x \). So \( Py = PP^T x, x = Py \) and \( P = \left[ \begin{array}{cc} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{array} \right] \).

(c) \( 8x_1^2 + 6x_1x_2 \).

Let \( Q(x_1, x_2) = 8x_1^2 + 6x_1x_2 = x^T \left[ \begin{array}{cc} 8 & 3 \\ 3 & 0 \end{array} \right] x \) and \( A = \left[ \begin{array}{cc} 8 & 3 \\ 3 & 0 \end{array} \right] \). We want to orthogonally diagonalize \( A \).

Compute \( A - \lambda I = \left[ \begin{array}{cc} 8 - \lambda & 3 \\ 3 & 0 - \lambda \end{array} \right] \) and \( \det(A - \lambda I) = (8 - \lambda) \cdot (-\lambda - 9) = \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1) \). So \( \lambda = -1 \) or \( \lambda = 9 \). Since \( A \) has positive and negative eigenvalues, we know that the quadratic form is indefinite.

Now we diagonalize \( A \).

\( \lambda = -1: A - (-1) \cdot I = \left[ \begin{array}{cc} 8 - (-1) & 3 \\ 3 & 0 - (-1) \end{array} \right] = \left[ \begin{array}{cc} 9 & 3 \\ 3 & 1 \end{array} \right] \).

So \( x \in \text{Null}(A - I) \) iff \( 3x_1 + x_2 = 0 \). So \( x_2 = -3x_1 \) and \( x = \left[ \begin{array}{c} x_1 \\ -3x_1 \end{array} \right] = x_1 \left[ \begin{array}{c} 1 \\ -3 \end{array} \right] \). So \( \left[ \begin{array}{c} 1 \\ -3 \end{array} \right] \) is an eigenvector corresponding to eigenvalue \( \lambda = -1 \).

\( \lambda = 9: A - 9 \cdot I = \left[ \begin{array}{cc} 8 - 9 & 3 \\ 3 & 0 - 9 \end{array} \right] = \left[ \begin{array}{cc} -1 & 3 \\ 3 & -9 \end{array} \right] \).

So \( x \in \text{Null}(A - 9 \cdot I) \) iff \( x_1 - 3x_2 = 0 \). So \( x_1 = 3x_2 \) and \( x = \left[ \begin{array}{c} 3x_2 \\ x_2 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 1 \end{array} \right] \).

So \( \left[ \begin{array}{c} 3 \\ 1 \end{array} \right] \) is an eigenvector corresponding to eigenvalue \( \lambda = 9 \).

Now \( \{v_1 = \left[ \begin{array}{c} 1 \\ -3 \end{array} \right], v_2 = \left[ \begin{array}{c} 3 \\ 1 \end{array} \right] \} \) is an orthogonal basis.
\[ \|v_1\| = \sqrt{10} \quad \text{and} \quad \|v_2\| = \sqrt{10}. \] Thus \( \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{10}} \frac{1}{\sqrt{10}} \), \( \frac{v_2}{\|v_2\|} = \frac{3}{\sqrt{10}} \frac{1}{\sqrt{10}} \) is an orthonormal basis of eigenvectors. So we have

\[ A = P \begin{bmatrix} -1 & 0 & 9 \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} P^T \]

Now \( Q(x) = x^T A x = x^T P \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^T x = y^T \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} y = -y_1^2 + 9y_2^2 \) if \( y = P^T x \). So \( P y = P P^T x, x = P y \) and \( P \begin{bmatrix} -1 & 0 & 9 \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \).

3. (a) Find a \( 3 \times 3 \) matrix \( A \) which is not diagonalizable?

Solution: Let \( A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \). Then \( \det(A - \lambda I) = -\lambda^3 \) and the eigenvalues of \( A \) are zero.

\[
A - 0 \cdot I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

The eigenvector \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) satisfies \( x_2 = 0 \) and \( x_3 = 0 \). The eigenvector is \( x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \). So there is only one eigenvector for \( A \) and \( A \) is not diagonalizable.

(b) Give an example of a \( 2 \times 2 \) matrix which is diagonalizable but not orthogonally diagonalizable?

Solution: Let \( A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \). Then \( \det(A - \lambda I) = \begin{bmatrix} 1 - \lambda & 4 \\ 1 & 1 - \lambda \end{bmatrix} \) = \((1 - \lambda)^2 - 4 = (1 - \lambda)^2 - 2^2 = (1 - \lambda - 2)(1 - \lambda + 2) = (-\lambda - 1)(3 - \lambda)\). So \( A \) has two distinct eigenvalues and \( A \) is diagonalizable. But \( A \) is not symmetric. So \( A \) is not orthogonally diagonalizable.
4. Let \( A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \).

(a) Find the condition on \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) such that \( Ax = b \) is solvable.

Solution:

Consider the augmented matrix \( [A \ b] = \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 1 & 1 & 0 & b_2 \\ 0 & 1 & 2 & b_3 \\ -1 & 0 & -1 & b_4 \end{bmatrix} \)

\( a_2 := \tilde{a}_2 + (-1)a_1 \)

\( a_4 := \tilde{a}_4 + a_1 \)

\( a_2 := -\tilde{a}_2 \)

\( a_3 := a_3 - \tilde{a}_2, a_4 := a_4 - 2a_2 \)

8
\[
\begin{bmatrix}
1 & 2 & 2 & b_1 \\
0 & 1 & 2 & -b_2 + b_1 \\
0 & 0 & -3 & b_4 - b_1 + 2b_2 \\
0 & 0 & 0 & b_3 + b_2 - b_1
\end{bmatrix}
\]

From here, we can see that \( Ax = b \) has a solution if \( b_3 + b_2 - b_1 = 0 \).

(b) What is the column space of \( A \)?

Solution:

The column space is the subspace spanned by the column vectors.

From the computation in (a), we know that the column vectors of \( A \) are independent. So \( \text{Col}(A) = \text{span}\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \} \).

(c) Describe the subspace \( \text{col}(A) \perp \) and find an basis for \( \text{col}(A) \perp \).

Solution: \( \text{col}(A) \perp = \{ x \mid x \cdot y = 0 \text{ for all } y \in \text{col}(A) \} \)

\[
\begin{align*}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} &= 0, \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} &= 0, \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} &= 0
\end{align*}
\]

\[
\begin{align*}
\{ x_1 + x_2 - x_4 = 0, 2x_1 + x_2 + x_3 = 0, 2x_1 + 2x_3 - x_4 = 0 \}
\end{align*}
\]

Consider

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 0 & -1 \\
2 & 1 & 1 & 0 \\
2 & 0 & 0 & -1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
r_2 := \begin{bmatrix}
1 & 1 & 0 & -1 \\
0 & 1 & 1 \\
2 & 0 & 2 & -1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
r_3 := \begin{bmatrix}
1 & 1 & 0 & -1 \\
0 & 1 & 1 \\
0 & -2 & 2 & 1
\end{bmatrix}
\end{align*}
\]
(d) Use Gram-Schmidt process to find an orthogonal basis for the column of the matrix $A$.

Solution:

Let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $w_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

Gram-Schmidt process is

$v_1 = w_1$, $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$ and $v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2$.

So $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

Compute $v_3 \cdot v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$, $v_3 \cdot v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = 3$, $v_3 \cdot v_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = 3$.

$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$ and

$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

So $x_1 + x_3 = 0$, $x_2 - x_3 = 0$ and $x_4 = 0$, $x_3$ is free. This implies that $x_1 = -x_3$, $x_2 = x_3$, $x_4 = 0$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Hence $\text{col}(A)^\perp = \text{span}\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\}$ and $\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\}$ is a basis for $\text{col}(A)^\perp$.

The dimension of $\text{col}(A)^\perp$ is 1.
(e) Find an orthonormal basis for the column of the matrix $A$.

Solution:

Note that $||v_1|| = \sqrt{v_1 \cdot v_1} = \sqrt{3}$, $||v_2|| = \sqrt{v_2 \cdot v_2} = \sqrt{3}$ and $||v_3|| = \sqrt{v_3 \cdot v_3} = \sqrt{3}$. Hence \( \{v_1, v_2, v_3\} \) is an orthonormal basis for $Col(A)$.

(f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of $A$ and write $y = \hat{y} + z$ where $\hat{y} \in col(A)$ and $z \in col(A)^\perp$. Also find the shortest distance from $y$ to $Col(A)$.

Solution: Since \( \{v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \} \) is an orthogonal basis for $Col(A)$, $y = \hat{y} + z$ where $\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \in Col(A)$ and $z = y - \hat{y} \in Col(A)^\perp$. Compute

\[
y \cdot v_1 = \begin{bmatrix} 3 \\ -2 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 7 + 3 + 0 + 2 = 12,
v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 1 + 1 = 3,
y \cdot v_2 = \begin{bmatrix} 3 \\ 10 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 7 + 0 + 10 - 2 = 15,
v_2 \cdot v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 3,
y \cdot v_3 = \begin{bmatrix} 3 \\ 10 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0 - 3 + 10 + 2 = 9,
v_3 \cdot v_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = 3.
\]
So \( \hat{y} = \frac{12}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{15}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{9}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4+5+0 \\ 4+0-3 \\ 0+5+3 \\ -4+5-3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -2 \\ 0 \end{bmatrix} \) and

\[
\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}.
\]

Note that \( z \in \text{Col}(A)^\perp = \text{span}\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \} \).

The shortest distance from \( y \) to \( \text{Col}(A) \) is \( \|y - \hat{y}\| = \|z\| = \sqrt{2^2 + (-2)^2 + 2^2 + 0^2} = \sqrt{12} \).

(g) Using previous result to explain why \( Ax = y \) has no solution.
Solution: Since the orthogonal projection of \( y \) to \( \text{Col}(A) \) is not \( y \), this implies that \( y \) is not in \( \text{Col}(A) \). So \( Ax = y \) has no solution.

(h) Use orthogonal projection to find the least square solution of \( Ax = y \).
Solution: The least square solution of \( Ax = y \) is the solution of \( Ax = \hat{y} = \begin{bmatrix} 9 \\ 1 \\ -2 \\ 0 \end{bmatrix} \) where \( \hat{y} \) is the orthogonal projection of \( y \) onto the column space of \( A \) (from part (f), we know \( \hat{y} = \begin{bmatrix} 9 \\ 1 \\ -2 \\ 0 \end{bmatrix} \)).

Consider the augmented matrix

\[
[A \hat{y}] = \begin{bmatrix}
1 & 2 & 2 & | & 9 \\
1 & 1 & 0 & | & 1 \\
0 & 1 & 2 & | & 8 \\
-1 & 0 & -1 & | & -2
\end{bmatrix}
\]

\[
r_2 := r_2 - r_1, r_3 := r_3 + r_1
\]

\[
\begin{bmatrix}
1 & 2 & 2 & | & 9 \\
0 & -1 & -2 & | & -8 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & -3 & | & -9
\end{bmatrix}
\]

\[
r_3 := r_3 + r_2, r_4 := r_4 + r_1
\]

\[
\begin{bmatrix}
1 & 2 & 2 & | & 9 \\
0 & 1 & 2 & | & 8 \\
0 & 0 & 1 & | & 3 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]

12
\[ r_2 := r_2 - 2r_3, \quad r_1 := r_1 - 2r_3 \]

\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ r_1 := r_1 - 2r_2 \]

So \( x_1 = -1, \ x_2 = 2, \ x_3 = 3 \) and the least square solution of \( Ax = y \) is \( x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \).

(i) Use normal equation to find the least square solution of \( Ax = y \).

Solution: The normal equation is \( A^T Ax = A^T y \). Compute \( A^T A = \)

\[
\begin{bmatrix}
3 & 3 & 3 \\
3 & 6 & 6 \\
3 & 6 & 9
\end{bmatrix}
\]

and \( A^T y = \begin{bmatrix} 12 \\ 27 \\ 36 \end{bmatrix} \).

So the normal equation \( A^T Ax = A^T y \) is

\[
\begin{bmatrix}
3 & 3 & 3 & 12 \\
3 & 6 & 6 & 27 \\
3 & 6 & 9 & 36
\end{bmatrix}
\]

Consider the augmented matrix

\[
\begin{bmatrix}
3 & 3 & 3 & 12 \\
3 & 6 & 6 & 27 \\
3 & 6 & 9 & 36
\end{bmatrix}
\]
\[ r_2 := r_2 - r_1, r_3 := r_3 - r_1 \begin{bmatrix} 3 & 3 & 3 \mid 12 \\ 0 & 3 & 3 \mid 15 \\ 0 & 3 & 6 \mid 24 \end{bmatrix} \]

\[ \sim r_3 := r_3 - r_2 \begin{bmatrix} 3 & 3 & 3 \mid 12 \\ 0 & 3 & 3 \mid 15 \\ 0 & 0 & 3 \mid 9 \end{bmatrix} \sim r_1 := r_1 / 3, r_2 := r_2 / 3, r_3 := \]

\[ \begin{bmatrix} 1 & 1 & 1 \mid 4 \\ 0 & 1 & 1 \mid 5 \\ 0 & 0 & 1 \mid 3 \end{bmatrix} \]

\[ \sim r_2 := r_2 - r_3, r_1 := r_1 - r_3 \begin{bmatrix} 1 & 1 & 0 \mid 1 \\ 0 & 1 & 0 \mid 2 \\ 0 & 0 & 1 \mid 3 \end{bmatrix} \]

\[ \sim r_1 := r_1 - r_2, \begin{bmatrix} 1 & 0 & 0 \mid -1 \\ 0 & 1 & 0 \mid 2 \\ 0 & 0 & 1 \mid 3 \end{bmatrix} \]

So \( x_1 = -1, x_2 = 2, x_3 = 3 \) and the least square solution of \( Ax = y \) is \( x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \).

5. Find the equation \( y = a + mx \) of the least square line that best fits the given data points. \((0, 1), (1, 1), (3, 2)\).

Solution: We try to solve the equations \( 1 = a, 1 = a + m, 2 = a + 3m \), that is,
\( a = 1, a + m = 1 \) and \( a + 3m = 2 \). It corresponding to the linear system
\[
\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
\]

Let \( A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \). We solve the normal equation
The solution includes calculations and explanations for various mathematical problems, including:

1. Computing the transpose of a matrix and multiplying it by itself.
2. Computing the transpose of a matrix and then multiplying it by another matrix.
3. Considering an augmented matrix and performing row operations to find the least square solution.
4. Solving a system of linear equations to find the least square line that best fits the given data points.
5. Showing that a set of vectors forms an orthonormal basis.
6. Finding the coordinates of a vector with respect to a given basis.
\[-\frac{3}{5} - \frac{4}{5} = -\frac{7}{5}, \quad y \cdot u_2 = (1, -1, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}, \quad y \cdot u_3 = (1, -1, 2) \cdot (0, 0, 1) = 2.

So the coordinate of $y$ with respect to the basis in (a) is $(-\frac{7}{5}, \frac{1}{5}, 2)$.

7. (a) Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Find the inverse matrix of $A$ if possible.

Solution: Consider the augmented matrix $[AI] = \begin{bmatrix} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$.

$r_1 := r_1 - r_3 \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$

$r_3 := r_3 - 2r_1 \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{bmatrix}$

$r_2 := r_2 + r_3 \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & -1 & -1 & -2 & 1 & 3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{bmatrix}$

$r_3 := r_3 + 3r_2 \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{bmatrix}$

$r_2 := r_2 - r_3, r_1 := r_1 - 3r_3 \begin{bmatrix} 1 & 3 & 0 & -11 & 9 & 17 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{bmatrix}$

$r_1 := r_1 - 3r_2 \begin{bmatrix} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{bmatrix}$.
So \( A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} \).

(b) Find the coordinates of the vector \( (1, -1, 2) \) with respect to the basis \( B \) obtained from the column vectors of \( A \).

Solution: The coordinate is \( x = A^{-1} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -5 \end{bmatrix} \).

8. Let \( H = \left\{ \begin{bmatrix} a + 2b - c \\ a - b - 4c \\ a + b - 2c \end{bmatrix} : a, b, c \text{ any real numbers} \right\} \).

a. Explain why \( H \) is a subspace of \( \mathbb{R}^3 \).

Solution: \( \begin{bmatrix} a + 2b - c \\ a - b - 4c \\ a + b - 2c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \)

So \( H = \text{Span} \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \} \) and \( H \) is a subspace.

b. Find a set of vectors that spans \( H \).

Solution: \{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ -4 \\ -2 \end{bmatrix} \} \) spans the space \( H \).

c. Find a basis for \( H \).

Solution: Consider the matrix \( A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & -4 \\ 1 & 1 & -2 \end{bmatrix} \)

\( r_2 := r_2 - r_1, r_3 := r_3 - r_1 \)

\( \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -3 \\ 0 & -1 & -1 \end{bmatrix} \)
\[ r_2 := \frac{r_2}{(-3)} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad r_3 := r_3 + r_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \]

So the first two vectors are pivot vectors and \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \) is a basis.

The dimension of the subspace is 2.

d. What is the dimension of the subspace?
Solution: The dimension of the subspace is 2.

e. Find an orthogonal basis for \( H \).
Solution: Let \( u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \).

Then \( v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 \). Compute \( u_2 \cdot v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 - 1 + 1 = 2 \) and \( v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 + 1 + 1 = 3 \).

\[ v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{-5}{3} \\ \frac{1}{3} \end{bmatrix}. \]

Thus \( \{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{4}{3} \\ \frac{-5}{3} \\ \frac{1}{3} \end{bmatrix} \} \) is an orthogonal basis for \( H \). We can verify that \( v_1 \cdot v_2 = 0 \).

9. Determine if the following systems are consistent and if so give all solutions in parametric vector form.
(a)

\[
\begin{align*}
x_1 - 2x_2 &= 3 \\
2x_1 - 7x_2 &= 0 \\
-5x_1 + 8x_2 &= 5
\end{align*}
\]

Solution: The augmented matrix is
\[
\begin{bmatrix} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_2 := r_2 - 2r_1)
\]

18
The augmented matrix is
\[
\begin{bmatrix}
1 & 2 & -3 & 1 & 1 \\
0 & 0 & 1 & 0 & 7 \\
-2 & -4 & 7 & -1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -3 & 1 & 1 \\
0 & 0 & 1 & 0 & 7 \\
-2 & -4 & 7 & -1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -3 & 1 & 1 \\
0 & 0 & 1 & 0 & 7 \\
0 & 0 & 1 & 1 & 3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -3 & 1 & 1 \\
0 & 0 & 1 & 0 & 7 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -3 & 0 & 5 \\
0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & 0 & 26 \\
0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\]
So \(x_2\) is free. The solution is \(x_1 = 26 - 2x_2, x_3 = 7, x_4 = -47\). Its
parametric vector form is
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26 - 2x_2 \\ x_2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.
\]

10. Let
\[
A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}
\]
which is row reduced to
\[
\begin{bmatrix} 1 & -3 & -2 & -20 & -3 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

(a) Find a basis for the column space of \( A \)
(b) Find a basis for the nullspace of \( A \)
(c) Find the rank of the matrix \( A \)
(d) Find the dimension of the nullspace of \( A \).

(e) Is \[
\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}
\] in the range of \( A \)?

(e) Does \( Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} \) have any solution? Find a solution if it’s solvable.

Solution: Consider the augmented matrix
\[
\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 2 & -6 & 9 & -1 & 8 & 4 & 3 \\ 2 & -6 & 9 & -1 & 9 & 3 & 2 \\ -1 & 3 & -4 & 2 & -5 & 1 & 0 \end{bmatrix}
\]
\[-2r_1 + r_2, \overbrace{-2r_1 + r_3}^{r_3}, r_1 + r_4 \]
\[
\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}
\]
So the first, third and fifth vector forms a basis for Col(A), i.e \{ 
\begin{pmatrix}
1 \\
3 \\
2 \\
-1 \\
0
\end{pmatrix}, 
\begin{pmatrix}
4 \\
9 \\
9 \\
-1 \\
1
\end{pmatrix}, 
\begin{pmatrix}
5 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\} is a basis for Col(A). The rank of \( A \) is 3 and the dimension of the null space is \( 5 - 3 = 2 \).

\( x \in Null(A) \) if \( x_1 - 3x_2 - 14x_4 = 0, x_3 + 3x_4 = 0 \) and \( x_5 = 0 \). So
\[
x = \begin{bmatrix}
3x_2 + 14x_4 \\
x_2 \\
x_4 \\
0
\end{bmatrix} = \begin{bmatrix}
3 \\
1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
14 \\
0 \\
-1 \\
0
\end{bmatrix} \cdot x_4 \cdot \begin{bmatrix}
3 \\
1 \\
0 \\
0
\end{bmatrix} \cdot \begin{bmatrix}
14 \\
0 \\
-1 \\
0
\end{bmatrix}.
\]
Thus \( \{ \begin{bmatrix}
1 \\
2 \\
2 \\
-1
\end{bmatrix}, \begin{bmatrix}
4 \\
9 \\
9 \\
-4
\end{bmatrix}, \begin{bmatrix}
5 \\
8 \\
9 \\
-5
\end{bmatrix}\} \) is a basis for \( NULL(A) \).

From the result of row reduction, we can see that \( Ax = \begin{bmatrix}
1 \\
4 \\
3 \\
1
\end{bmatrix} \) is incon-
sistent (not solvable) and \[
\begin{bmatrix}
1 \\
4 \\
3 \\
1
\end{bmatrix}
\] is not in the range of \(A\).

From the result of row reduction, we can see that \(Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}\) is solvable.

11. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

\[
\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
1 \\
-2 \\
4
\end{bmatrix}, \begin{bmatrix}
-4 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
-4 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
-4 \\
-1 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
4
\end{bmatrix}, \begin{bmatrix}
-3 \\
-1 \\
4
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
5 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
6 \\
3 \\
3
\end{bmatrix}
\]

Solution: \(\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 - 1 = 1 \neq 0\). So the columns of the matrix form a linearly independent set.

\[
\begin{bmatrix}
1 \\
-2 \\
3 \\
4
\end{bmatrix} \] . The second column vector is a multiple of the first column vector. So the columns of the matrix form a linearly dependent set.

\[
\begin{bmatrix}
-4 \\
0 \\
1 \\
5
\end{bmatrix}, \begin{bmatrix}
-3 \\
-1 \\
0 \\
4
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
-3 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
3 \\
12 \\
7
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
-4 \\
0
\end{bmatrix}
\]

\[
\text{interchange first and third row}
\]

\[
\begin{bmatrix}
1 \\
0 \\
3 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
4 \\
-4
\end{bmatrix}, \begin{bmatrix}
3 \\
-1 \\
4 \\
-3
\end{bmatrix}, \begin{bmatrix}
1 \\
5 \\
4 \\
6
\end{bmatrix}
\]

\[
\text{interchange third and fourth row; } \frac{1}{7}r_4
\]

\[
\begin{bmatrix}
1 \\
0 \\
3 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
-4 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

22
This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

The column vectors of

\[
\begin{bmatrix}
-4 & -3 & 1 & 5 & 1 \\
2 & -1 & 4 & -1 & 2 \\
1 & 2 & 3 & 6 & -3 \\
5 & 4 & 6 & -3 & 2
\end{bmatrix}
\]

form a dependent set since we have five column vectors in \( \mathbb{R}^4 \).

12. Let \( A \) be a \( 12 \times 5 \) matrix. You may assume that \( \text{Nul}(A^T A) = \text{Nul}(A) \). (This relation holds for any matrix \( A \).)

a. What is the size of \( A^T A \)?

b. Use the Rank Theorem to obtain an equation involving \( \text{rank} A \). Find another equation involving \( \text{rank}(A^T A) \). What is the connection between these two ranks?

c. Suppose the columns of \( A \) are linearly independent. Explain why \( A^T A \) is invertible.

Solution: a. Note that \( \text{Nul}(A) \) is the dimension of the null space of \( A \). Since \( A^T \) is a \( 5 \times 12 \) matrix and \( A \) is a \( 12 \times 5 \) matrix, we know that \( A^T A \) is a \( 5 \times 5 \) matrix.

b. \( \text{rank}(A) + \text{Nul}(A) = 5 \) and \( \text{rank}(A^T A) + \text{Nul}(A^T A) = 5 \). Using the fact that \( \text{Nul}(A^T A) = \text{Nul}(A) \), we know that \( \text{rank}(A) = \text{rank}(A^T A) \).

c. The columns of \( A \) are linearly independent implies that \( \text{rank}(A) = 5 \). So \( \text{rank}(A^T A) = 5 \). Recall that \( A^T A \) is a \( 5 \times 5 \) matrix. This implies that \( A^T A \) is an invertible matrix.