**Linear Algebra (Math 2890) Solution to Final Review Problems**

1. Let $A$ be the matrix
$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$
.

   (a) Prove that $det(A - \lambda I) = (1-\lambda)^2(4-\lambda)$.

   Solution: Compute $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$ and
   $det(A - \lambda I) = (2-\lambda)^3 + 1 + 1 - (2-\lambda) - (2-\lambda) - (2-\lambda) = 8 - 12\,\lambda + 6\,\lambda^2 - \lambda^3 + 2 - 6 + 3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (1-\lambda)^2(4-\lambda)$.

   (b) Orthogonally diagonalizes the matrix $A$, giving an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^t$

   Solution: We know that the eigenvalues are 1 ,1 and 4.

   When $\lambda = 1$, $A - (1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   $x \in Null(A - I)$ if $x_1 + x_2 + x_3 = 0$. So $x_1 = -x_2 - x_3$ and
   $x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Thus $\{w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis of eigenvectors when $\lambda = -1$.

   Now we use Gram-Schmidt process to find an orthogonal basis for $Null(A - I)$.

   Let $v_1 = w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $w_2 \cdot v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1$ and $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2$.

   So $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - (\frac{1}{2}) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

Hence $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\}$is an orthogonal basis for $Null(A - I)$.

When $\lambda = 4$, $A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ *interchange* $r_1$ *and* $r_2$,

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$-2\ r_1 + r_2, -r_1 + r_3 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$

$r_2 + r_3, r_2/(-3) \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} 2r_2 + r_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x \in Null(A - 4I)$ if $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$. So $x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $\{v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ is a basis for $Null(A - 4I)$.

So $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ is an orthogonal basis for $R^3$ which are eigenvectors corresponding to $\lambda = 1$, $\lambda = 1$ and $\lambda = 4$. Compute $||v_1|| = \sqrt{2}$, $||v_2|| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$ and $||v_3|| = \sqrt{3}$.

Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \frac{v_3}{||v_3||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\}$ is an orthonormal basis for $R^3$ which are eigenvectors corresponding to

$\lambda = 1$, $\lambda = 1$ and $\lambda = 4$.

Finally, we have $A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} P^T$ where $P = [\frac{v_1}{||v_1||} \; \frac{v_2}{||v_2||} \; \frac{v_3}{||v_3||}] = \begin{bmatrix} \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

(c) Write the quadratic form associated with $A$ using variables $x_1$, $x_2$, and $x_3$?

Solution: Recall that the quadratic form in $x_1, x_2$ and $x_3$ is $Q_A(x) = x^T Ax = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

(d) Find $A^{-1}$, $A^{10}$ and $e^A$.

Solution:Recall that $A = PDP^T$. Then $A^{-1} = PD^{-1}P^T = P \begin{bmatrix} 1^{-1} & 0 & 0 \\ 0 & 1^{-1} & 0 \\ 0 & 0 & 4^{-1} \end{bmatrix} P^T = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} P^T$.

$A^{10} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{10} \end{bmatrix} P^T$ and $e^A = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^T$

(e) What's $A^{-5}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix})$?

Solution: Note that $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 4. So we have $A(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A^k(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 4^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence $A^{-5}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 4^{-5} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(f) What is $lim_{n\to\infty} A^{-n}$? Recall that $A = PDP^T$ and $A^{-n} = PD^{-n}P^T = P \begin{bmatrix} 1^{-n} & 0 & 0 \\ 0 & 1^{-n} & 0 \\ 0 & 0 & 4^{-n} \end{bmatrix} P^T = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{-n} \end{bmatrix} P^T$. Note that $\lim_{n\to\infty} 4^{-n} =$

0. So we have $\lim_{n\to\infty} A^{-n} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T$.

2. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

   (a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.

   Let $Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2 = x^T \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} x$ and $A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$. We want to orthogonally diagonalizes $A$.

   Compute $A - \lambda I = \begin{bmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (9-\lambda)(3-\lambda) - 16 = \lambda^2 - 12\lambda + 27 - 16 = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11)$. So $\lambda = 1$ or $\lambda = 11$. Since the eigenvalues of $A$ are all positive, we know that the quadratic form is positive definite.

   Now we diagonalize $A$.

   $\lambda = 1$: $A - 1 \cdot I = \begin{bmatrix} 9-1 & -4 \\ -4 & 3-1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 1 \cdot I)$ iff $2x_1 - x_2 = 0$. So $x_2 = 2x_1$ and $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$.

   $\lambda = 11$: $A - 11 \cdot I = \begin{bmatrix} 9-11 & -4 \\ -4 & 3-11 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 11 \cdot I)$ iff $x_1 + 2x_2 = 0$. So $x_1 = -2x_2$ and $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 11$.

   Now $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1|| = \sqrt{5}$ and $||v_2|| = \sqrt{5}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T$ where $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Now $Q(x) = x^T A x = x^T P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T x = y^T \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} y = y_1^2 + 11y_2^2$ if $y = P^T x$. So $Py = PP^T x$, $x = Py$ and $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Note that we have used the fact that $PP^T = I$.

(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.

Let $Q(x_1, x_2) = -5x_1^2 + 4x_1x_2 - 2x_2^2 = x^T \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} x$ and $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. We want to orthogonally diagonalizes $A$.

Compute $A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (-5 - \lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+1)(\lambda+6)$. So $\lambda = -1$ or $\lambda = -6$. Since the eigenvalues of $A$ are all negative, we know that the quadratic form is negative definite.

Now we diagonalize $A$.

$\lambda = -1$: $A - (-1) \cdot I = \begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$.
So $x \in Null(A - 1 \cdot I)$ iff $2x_1 - x_2 = 0$. So $x_2 = 2x_1$ and $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$.

$\lambda = -6$: $A - (-6) \cdot I = \begin{bmatrix} -5-(-6) & 2 \\ 2 & (-2)-(-6) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.
So $x \in Null(A - 11 \cdot I)$ iff $x_1 + 2x_2 = 0$. So $x_1 = -2x_2$ and $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -6$.

Now $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute

$||v_1|| = \sqrt{5}$ and $||v_2|| = \sqrt{5}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = P \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} P^T$ where $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

Now $Q(x) = x^T Ax = x^T P \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} P^T x = y^T \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} y = -y_1^2 - 6y_2^2$ if $y = P^T x$. So $Py = PP^T x$, $x = Py$ and $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.

(c) $8x_1^2 + 6x_1x_2$.

Let $Q(x_1, x_2) = 8x_1^2 + 6x_1x_2 = x^T \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix} x$ and $A = \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}$. We want to orthogonally diagonalizes $A$.

Compute $A - \lambda I = \begin{bmatrix} 8-\lambda & 3 \\ 3 & 0-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (8-\lambda) \cdot (-\lambda) - 9 = \lambda^2 - 8\lambda - 9 = (\lambda + 1)(\lambda - 9)$. So $\lambda = -1$ or $\lambda = 9$. Since $A$ has positive and negative eigenvalues, we know that the quadratic form is indefinite.

Now we diagonalize $A$.

$\lambda = -1$: $A - (-1) \cdot I = \begin{bmatrix} 8-(-1) & 3 \\ 3 & 0-(-1) \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 1 \cdot I)$ iff $3x_1 + x_2 = 0$. So $x_2 = -3x_1$ and $x = \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -1$.

$\lambda = 9$: $A - 9 \cdot I = \begin{bmatrix} 8-9 & 3 \\ 3 & 0-9 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 9 \cdot I)$ iff $x_1 - 3x_2 = 0$. So $x_1 = 3x_2$ and $x = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 9$.

Now $\{v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute

$||v_1|| = \sqrt{10}$ and $||v_2|| = \sqrt{10}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = P \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^T$ where $P = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$.

Now $Q(x) = x^T Ax = x^T P \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^T x = y^T \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} y = -y_1^2 + 9y_2^2$ if $y = P^T x$. So $Py = PP^T x$, $x = Py$ and $P \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$.

3. (a) Find a $3 \times 3$ matrix $A$ which is not diagonalizable?

Solution: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then $det(A - \lambda I) = -\lambda^3$ and the eigenvalues of $A$ are zero.

$A - 0 \cdot I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies $x_2 = 0$ and $x_3 = 0$. The eigenvector is $x == \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So there is only one eigenvector for $A$ and $A$ is not diagonalizable.

(b) Give an example of a $2 \times 2$ matrix which is diagonalizable but not orthogonally diagonalizable?

Solution: Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. Then $det(A - \lambda I) == \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = (1-\lambda)^2 - 2^2 = (1-\lambda-2)(1-\lambda+2) = (-\lambda-1)(3-\lambda)$. So $A$ has two distinct eigenvalues and $A$ is diagonalizable. But $A$ is not symmetric. So $A$ is not orthogonally diagonalizable.

4. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.

(a) Find the condition on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $Ax = b$ is solvable.

Solution:

Consider the augmented matrix $[A \ \ b] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 1 & 1 & 0 & b_2 \\ 0 & 1 & 2 & b_3 \\ -1 & 0 & -1 & b_4 \end{array}\right]$

$a_2 := a_2 + (-1)a_1 \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -1 & -2 & b_2 - b_1 \\ 0 & 1 & 2 & b_3 \\ -1 & 0 & -1 & b_4 \end{array}\right]$

$a_4 := a_4 + a_1 \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -1 & -2 & b_2 - b_1 \\ 0 & 1 & 2 & b_3 \\ 0 & 2 & 1 & b_4 + b_1 \end{array}\right]$

$a_2 := -a_2 \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 1 & 2 & b_3 \\ 0 & 2 & 1 & b_4 + b_1 \end{array}\right]$

$a_3 := a_3 - a_2, a_4 := a_4 - 2a_2 \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \\ 0 & 0 & -3 & b_4 - b_1 + 2\,b_2 \end{array}\right]$

$$\widetilde{a_3 \leftrightarrow a_4}\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 0 & -3 & b_4 - b_1 + 2\,b_2 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{array}\right]$$

From here, we can see that $Ax = b$ has a solution if $b_3 + b_2 - b_1 = 0$.

(b) What is the column space of $A$?

Solution:

The column space is the subspace spanned by the column vectors. From the computation in (a), we know that the column vectors of $A$ are independent. So $Col(A) = span\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}\}$.

(c) Describe the subspace $col(A)^{\perp}$ and find an basis for $col(A)^{\perp}$.

Solution: $col(A)^{\perp} = \{x | x \cdot y = 0 \ for\ all\ y \in col(A)\}$

$$= \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 0\}$$

$$= \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | x_1 + x_2 - x_4 = 0, 2x_1 + x_2 + x_3 = 0, 2x_1 + 2x_3 - x_4 = 0\}$$

Consider $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \widetilde{r_2 := r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{bmatrix}$

$$\widetilde{r_3 := r_3 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & 2 & 1 \end{bmatrix} \widetilde{r_2 := -r_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 2 & 1 \end{bmatrix}$$

$\widetilde{r_3 := r_3 + 2r_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \widetilde{r_1 := r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

$\widetilde{r_3 := r_3/(-3)} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \widetilde{r_1 := r_1 - r_3, r_2 := r_2 + 2r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So $x_1 + x_3 = 0$, $x_2 - x_3 = 0$ and $x_4 = 0$, $x_3$ is free. This implies that $x_1 = -x_3$, $x_2 = x_3$ , $x_4 = 0$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Hence $col(A)^{\perp} = span\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$ and $\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$ is a basis for $col(A)^{\perp}$.

The dimension of $col(A)^{\perp}$ is 1.

(d) Use Gram-Schmidt process to find an orthogonal basis for the column of the matrix $A$.

Solution:

Let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $w_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}$.

Gram-Schmidt process is

$v_1 = w_1$, $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$ and $v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2$.

So $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Compute $w_2 \cdot v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$, $v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$ and $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Compute $w_3 \cdot v_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3$, $w_3 \cdot v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3$,

$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3$ and

$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2-1-1 \\ 0-1-0 \\ 2-0-1 \\ -1+1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Hence $\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}\}$ is an orthogonal basis for $Col(A)$.

(e) Find an orthonormal basis for the column of the matrix $A$.
Solution:
Note that $||v_1|| = \sqrt{v_1 \cdot v_1} = \sqrt{3}$, $||v_2|| = \sqrt{v_2 \cdot v_2} = \sqrt{3}$ and $||v_3|| = \sqrt{v_3 \cdot v_3} = \sqrt{3}$. Hence $\{\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||}\} = \{\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}\}$ is an orthonormal basis for $Col(A)$.

(f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of $A$ and write $y = \widehat{y} + z$ where $\widehat{y} \in col(A)$ and $z \in col(A)^{\perp}$. Also find the shortest distance from $y$ to $Col(A)$.

Solution: Since $\{v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}\}$ is an orthogonal basis for $Col(A)$, $y = \widehat{y} + z$ where $\widehat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \in Col(A)$ and $z = y - \widehat{y} \in Col(A)^{\perp}$. Compute
$y \cdot v_1 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 7+3+0+2 = 12$, $v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 1 + 1 = 3$, $y \cdot v_2 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 7 + 0 + 10 - 2 = 15$,
$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3$,
$y \cdot v_3 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0-3+10+2 = 9$, $v_3 \cdot v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3$.

So $\widehat{y} = \frac{12}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{(15)}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \frac{9}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}4+5+0\\4+0-3\\0+5+3\\-4+5-3\end{bmatrix} = \begin{bmatrix}9\\1\\8\\-2\end{bmatrix}$ and $z = y - \widehat{y} = \begin{bmatrix}7\\3\\10\\-2\end{bmatrix} - \begin{bmatrix}9\\1\\8\\-2\end{bmatrix} = \begin{bmatrix}2\\-2\\2\\0\end{bmatrix}$. Note that $z \in Col(A)^{\perp} = span\{\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}\}$.

The shortest distance from $y$ to $Col(A) = ||y - \widehat{y}|| = ||z|| = \sqrt{(2)^2 + (-2)^2 + (2)^2 + (0)^2} = \sqrt{12}$.

(g) Using previous result to explain why $Ax = y$ has no solution.
Solution: Since the orthogonal projection of $y$ to $Col(A)$ is not $y$, this implies that $y$ is not in $Col(A)$. So $Ax = y$ has no solution.

(h) Use orthogonal projection to find the least square solution of $Ax = y$.
Solution: The least square solution of $Ax = y$ is the solution of $Ax = \widehat{y} = \begin{bmatrix}9\\1\\8\\-2\end{bmatrix}$ where $\widehat{y}$ is the orthogonal projection of $y$ onto the column space of $A$ (from part (f), we know $\widehat{y} = \begin{bmatrix}9\\1\\8\\-2\end{bmatrix}$.)
Consider the augmented matrix

$$[A\,\widehat{y}] = \left[\begin{array}{ccc|c}1 & 2 & 2 & 9\\1 & 1 & 0 & 1\\0 & 1 & 2 & 8\\-1 & 0 & -1 & -2\end{array}\right] r_2 := r_2 - \widetilde{r_1, r_3} := r_3 + r_1 \left[\begin{array}{ccc|c}1 & 2 & 2 & 9\\0 & -1 & -2 & -8\\0 & 1 & 2 & 8\\0 & 2 & 1 & 7\end{array}\right]$$

$$r_3 := r_3 + \widetilde{r_2, r_4} := r_4 + r_1 \left[\begin{array}{ccc|c}1 & 2 & 2 & 9\\0 & -1 & -2 & -8\\0 & 0 & 0 & 0\\0 & 0 & -3 & -9\end{array}\right]$$

$$r_2 := -r_2, r_4 \widetilde{:= r_4}/(-3), r_3 \leftrightarrow r_4 \left[\begin{array}{ccc|c}1 & 2 & 2 & 9\\0 & 1 & 2 & 8\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{array}\right]$$

$$r_2 := r_2 - \widetilde{2r_3, r_1} := r_1 - 2r_3 \begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$r_1 := \widetilde{r_1} - 2r_2 \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

So $x_1 = -1$, $x_2 = 2$, $x_3 = 3$ and the least square solution of $Ax = y$ is $x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(i) Use normal equation to find the least square solution of $Ax = y$.
Solution: The normal equation is $A^T Ax = A^T y$. Compute $A^T A =$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 6 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

and $A^T y = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ 27 \\ 36 \end{bmatrix}$.

So the normal equation $A^T Ax = A^T y$ is

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 6 & 6 \\ 3 & 6 & 9 \end{bmatrix} x = \begin{bmatrix} 12 \\ 27 \\ 36 \end{bmatrix}.$$

Consider the augmented matrix $\begin{bmatrix} 3 & 3 & 3 & | & 12 \\ 3 & 6 & 6 & | & 27 \\ 3 & 6 & 9 & | & 36 \end{bmatrix} \sim$

$$r_2 := r_2 - r_1, r_3 := r_3 - r_1 \left[\begin{array}{ccc|c} 3 & 3 & 3 & 12 \\ 0 & 3 & 3 & 15 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

$$\sim r_3 := r_3 - r_2 \left[\begin{array}{ccc|c} 3 & 3 & 3 & 12 \\ 0 & 3 & 3 & 15 \\ 0 & 0 & 3 & 9 \end{array}\right] \sim r_1 := r_1/3, r_2 := r_2/3, r_3 :=$$

$$r_3/3 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim r_2 := r_2 - r_3,\ r_1 := r_1 - r_3 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim r_1 := r_1 - r_2, \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So $x_1 = -1$, $x_2 = 2$, $x_3 = 3$ and the least square solution of $Ax = y$ is $x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

5. Find the equation $y = a + mx$ of the least square line that best fits the given data points. $(0, 1)$, $(1, 1)$, $(3, 2)$.

   Solution: We try to solve the equations $1 = a$, $1 = a + m$, $2 = a + 3m$, that is,
   $a = 1$, $a+m = 1$ and $a+3m = 2$. It corresponding to the linear system
   $$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$
   Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$. We solve the normal equation

$$A^T A \begin{bmatrix} a \\ m \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Compute $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix}$ and

$A^T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$

Consider the augmented matrix

$\left[\begin{array}{cc|c} 3 & 4 & 4 \\ 4 & 10 & 7 \end{array}\right] \sim r_2 := r_2 - \frac{4}{3} r_1 \left[\begin{array}{cc|c} 3 & 4 & 4 \\ 0 & \frac{14}{3} & \frac{5}{3} \end{array}\right]$

$\sim r_2 := \frac{3}{14} r_2 \left[\begin{array}{cc|c} 3 & 4 & 4 \\ 0 & 1 & \frac{5}{14} \end{array}\right] \sim r_1 := r_1 - 4r_2 \left[\begin{array}{cc|c} 3 & 0 & \frac{18}{7} \\ 0 & 1 & \frac{5}{14} \end{array}\right]$

$\sim r_1 := r_1/3 \left[\begin{array}{cc|c} 1 & 0 & \frac{6}{7} \\ 0 & 1 & \frac{5}{14} \end{array}\right]$

So the least square solution is $a = \frac{6}{7}$ and $m = \frac{5}{14}$. The equation $y = \frac{6}{7} + \frac{5}{14} x$ is the least square line that best fits the given data points. $(0, 1)$, $(1, 1)$, $(3, 2)$.

6. (a) Show that the set of vectors

$$B = \left\{ u_1 = \left( -\frac{3}{5}, \frac{4}{5}, 0 \right), \ u_2 = \left( \frac{4}{5}, \frac{3}{5}, 0 \right), \ u_3 = (0, 0, 1) \right\}$$

is an **orthonormal basis** of $\mathbb{R}^3$.

Solution: Compute $u_1 \cdot u_2 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{-12}{5} + \frac{12}{5} = 0$, $u_1 \cdot u_3 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot (0, 0, 1) = 0, u_2 \cdot u_3 = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot (0, 0, 1) = 0$, $u_1 \cdot u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right) = \frac{9}{25} + \frac{16}{25} = 1$, $u_3 \cdot u_3 = (0, 0, 1) \cdot (0, 0, 1) = 1, u_2 \cdot u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{16}{25} + \frac{9}{25} = 1$

(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis in (a).

Solution: Let $y = (1, -1, 2)$. So $y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3 = (y \cdot u_1) u_1 + (y \cdot u_2) u_2 + (y \cdot u_3) u_3$. Compute $y \cdot u_1 = (1, -1, 2) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right) =$

$-\frac{3}{5} - \frac{4}{5} = -\frac{7}{5}$, $y \cdot u_2 = (1, -1, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$, $y \cdot u_3 = (1, -1, 2) \cdot (0, 0, 1) = 2$.

So the coordinate of $y$ with respect to the basis in (a) is $(-\frac{7}{5}, \frac{1}{5}, 2)$.

7. (a) Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Find the inverse matrix of $A$ if possible.

Solution: Consider the augmented matrix $[A\, I] = \left[\begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$\overset{\sim}{r_1 := r_1 - r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$\overset{\sim}{r_3 := r_3 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array}\right]$

$\overset{\sim}{r_2 := r_2 + r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & -1 & -1 & -2 & 1 & 3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array}\right] \overset{\sim}{r_2 := -r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array}\right]$

$\overset{\sim}{r_3 := r_3 + 3r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right]$

$\overset{\sim}{r_2 := r_2 - r_3, r_1 := r_1 - 3r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -11 & 9 & 17 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right]$

$\overset{\sim}{r_1 := r_1 - 3r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right].$

So $A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$.

(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis $B$ obtained from the column vectors of $A$.

Solution: The coordinate is $x = A^{-1} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} =$

$\begin{bmatrix} 8 \\ 2 \\ -5 \end{bmatrix}$.

8. Let $H = \left\{ \begin{bmatrix} a+2b-c \\ a-b-4c \\ a+b-2c \end{bmatrix} : a, b, c \text{any real numbers} \right\}$.

a. Explain why $H$ is a a subspace of $R^3$.

Solution: $\begin{bmatrix} a+2b-c \\ a-b-4c \\ a+b-2c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$

So $H = Span\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \}$ and $H$ is a subspace.

b. Find a set of vectors that spans $H$.

Solution:$\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix} \}$ spans the space $H$.

c. Find a basis for $H$.

Solution: Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$r_2 := r_2 - r_1, r_3 := r_3 - r_1 \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -3 \\ 0 & -1 & -1 \end{bmatrix}$

$$r_2 := \widetilde{r_2/(-3)} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} r_3 := \widetilde{r_3 + r_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the first two vectors are pivot vectors and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \}$ is a basis.

The dimension of the subspace is 2.

d. What is the dimension of the subspace?
Solution:The dimension of the subspace is 2.
e. Find an orthogonal basis for $H$.

Solution: Let $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Then $v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $u_2 \cdot v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 - 1 + 1 = 2$ and $v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 + 1 + 1 = 3$.
$v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$. Thus $\{v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{4}{3} \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}\}$ ia an orthogonal basis for $H$. We can verify that $v_1 \cdot v_2 = 0$.

9. Determine if the following systems are consistent and if so give all solutions in parametric vector form.
(a)

$$\begin{aligned} x_1 \quad -2x_2 &= 3 \\ 2x_1 \quad -7x_2 &= 0 \\ -5x_1 \quad +8x_2 &= 5 \end{aligned}$$

Solution: The augmented matrix is $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{bmatrix} \sim (r_2 := r_2 - 2r_1)$

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ -5 & 8 & 5 \end{bmatrix} \sim \ (r_3 := r_3 + 5r_1) \ \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & 20 \end{bmatrix}$
$\sim \ (r_2 := r_2/-3, r_3 := r_3/-2) \ \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -10 \end{bmatrix} \sim \ (r_3 := r_3 - r_2) \ \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$. The last row implies that $0 = -12$ which is impossible. So this system is inconsistent.

(b)

$$\begin{array}{lllll} x_1 & +2x_2 & -3x_3 & +x_4 & = 1 \\ -x_1 & -2x_2 & +4x_3 & -x_4 & = 6 \\ -2x_1 & -4x_2 & +7x_3 & -x_4 & = 1 \end{array}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim \ (r_2 := r_2 + r_1)$
$\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix} \sim \ (r_3 := r_3 + 2r_1) \ \begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$
$\sim (r_3 := r_3 - r_2) \ \begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 - r_3) \ \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$
$\sim (r_1 := r_1 - r_3) \ \begin{bmatrix} 1 & 2 & -3 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \sim (r_1 := r_1 + 3r_2) \ \begin{bmatrix} 1 & 2 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$.
So $x_2$ is free. The solution is $x_1 = 26 - 2x_2$, $x_3 = 7$, $x_4 = -47$. Its

parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26 - 2x_2 \\ x_2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

10. Let $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$ which is row reduced to $\begin{bmatrix} 1 & -3 & -2 & -20 & -3 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis for the column space of $A$
(b) Find a basis for the nullspace of $A$
(c) Find the rank of the matrix $A$
(d) Find the dimension of the nullspace of $A$.
(e) Is $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ in the range of $A$?

(e) Does $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ have any solution? Find a solution if it's solvable.

Solution: Consider the augmented matrix $\left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 2 & -6 & 9 & -1 & 8 & 4 & 3 \\ 2 & -6 & 9 & -1 & 9 & 3 & 2 \\ -1 & 3 & -4 & 2 & -5 & 1 & 0 \end{array}\right]$

$-2r_1 + r_2, \widetilde{-2r_1} + r_3, r_1 + r_4 \left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$

$$\widetilde{-r_2+r_3}\left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$2r_3+\widetilde{r_2, -5}r_3+r_1\left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 0 & 6 & 5 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\widetilde{-4r_2+r_1}\left[\begin{array}{ccccc|c|c} 1 & -3 & 0 & -14 & 0 & 6 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right].$$

So the first, third and fifth vector forms a basis for Col(A), i.e $\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}\right\}$ is a basis for Col(A). The rank of $A$ is 3 and the dimension of the null space is $5-3=2$.

$x \in Null(A)$ if $x_1 - 3x_2 - 14x_4 = 0$, $x_3 + 3x_4 = 0$ and $x_5 = 0$. So $x = \begin{bmatrix} 3x_2 + 14x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Thus$\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right.$ is a basis for $NULL(A)$.

From the result of row reduction, we can see that $Ax = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ is inconsistent (not solvable) and $\begin{bmatrix}1\\4\\3\\1\end{bmatrix}$ is not in the range of $A$.

From the result of row reduction, we can see that $Ax = \begin{bmatrix}0\\3\\2\\0\end{bmatrix}$ is solvable.

11. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}, \begin{bmatrix}1 & -2\\-2 & 4\\3 & 6\end{bmatrix}, \begin{bmatrix}-4 & -3 & 0\\0 & -1 & 4\\1 & 0 & 3\\5 & 4 & 6\end{bmatrix}, \begin{bmatrix}-4 & -3 & 1 & 5 & 1\\2 & -1 & 4 & -1 & 2\\1 & 2 & 3 & 6 & -3\\5 & 4 & 6 & -3 & 2\end{bmatrix}.$$

Solution: $det \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = 2 - 1 = 1 \neq 0$. So the columns of the matrix form a linearly independent set.

$\begin{bmatrix}1 & -2\\-2 & 4\\3 & 6\end{bmatrix}$. The second column vector is a multiple of the first column vector. So the columns of the matrix form a linearly dependent set.

$$\begin{bmatrix}-4 & -3 & 0\\0 & -1 & 4\\1 & 0 & 3\\5 & 4 & 6\end{bmatrix} \quad \widetilde{interchange\ first\ and\ third\ row} \quad \begin{bmatrix}1 & 0 & 3\\0 & -1 & 4\\-4 & -3 & 0\\5 & 4 & 6\end{bmatrix}$$

$$\widetilde{r_3 + 4r_1, r_4 + (-5)r_1} \begin{bmatrix}1 & 0 & 3\\0 & -1 & 4\\0 & -3 & 12\\0 & 4 & -9\end{bmatrix} \quad \widetilde{(-1)r_2} \begin{bmatrix}1 & 0 & 3\\0 & 1 & -4\\0 & -3 & 12\\0 & 4 & -9\end{bmatrix}$$

$$\widetilde{r_3 + 3r_2, r_4 + (-4)r_2} \begin{bmatrix}1 & 0 & 3\\0 & 1 & -4\\0 & 0 & 0\\0 & 0 & 7\end{bmatrix} \quad \widetilde{interchange\ 3rd\ and\ 4th\ row, \frac{1}{7}r_4} \begin{bmatrix}1 & 0 & 3\\0 & 1 & -4\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in $R^4$.

12. Let $A$ be a $12 \times 5$ matrix. You may assume that $Nul(A^TA) = Nul(A)$. (This relation holds form any matrix $A$.)
a. What is the size of $A^TA$?
b. Use the Rank Theorem to obtain an equation involving $rankA$. Find another equation involving $rank(A^TA)$. What is the connection between these two ranks?
c. Suppose the columns of $A$ are linearly independent. Explain why $A^TA$ is invertible.

Solution: a. Note that $Nul(A)$ is the dimension of the null space of $A$. Since $A^T$ is a $5 \times 12$ matrix and $A$ is a $12 \times 5$, we know that $A^TA$ is a $5 \times 5$ matrix.
b. $rank(A) + Nul(A) = 5$ and $rank(A^TA) + Nul(A^TA) = 5$. Using the fact that $Nul(A^TA) = Nul(A)$, we know that $rank(A) = rank(A^TA)$.
c. The columns of $A$ are linearly independent implies that $rank(A) = 5$. So $rank(A^TA) = 5$. Recall that $A^TA$ is a $5 \times 5$ matrix. This implies that $A^TA$ is a invertible matrix.