Linear Algebra (Math 2890) Review Problems for Final Exam

Final exam on Dec 14, Monday, 12:30pm-2:30pm.
Regular office hours:
UH2080B M 2:00-3:30pm W 11-12 a.m., 3-4 pm, F 2-3 pm
Office hour before the final exam:
Monday (Dec 7) 11-12, 2-3:30, Wednesday (Dec 9) 11-12, 3-4 and Friday (Dec 11)11-12, 2-3 p.m.
Monday (Dec 14) 10:30-12.
Topics in the final exam. The final exam is compressive. It coves 1.1 1.5, 1.7, 1.8, 2.1 2.3, 2.8, 2.9, 3.1, 3.2, 5.1 5.3, 6.1 6.6, 7.1, 7.2.

1. Let $A$ be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) Prove that $det(A - \lambda I) = (1 - \lambda)^2(4 - \lambda)$.

(b) Orthogonally diagonalizes the matrix $A$, giving an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^t$.

(c) Write the quadratic form associated with $A$ using variables $x_1$, $x_2$, and $x_3$.

(d) Find $A^{-1}$, $A^{10}$ and $e^A$.

(e) What’s $A^{-5}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

(f) What is $\lim_{n\to\infty}A^{-n}$?

2. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form in new variables $y_1, y_2$.

(a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.

(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.

(c) $8x_1^2 + 6x_1x_2$. 

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3. (a) Find a $3 \times 3$ matrix $A$ which is not diagonalizable?
   (b) Give an example of a $2 \times 2$ matrix which is diagonalizable but not orthogonally diagonalizable?

4. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.
   (a) Find the condition on $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $Ax = b$ is consistent.
   (b) What is the column space of $A$?
   (c) Describe the subspace $\text{col}(A)^\perp$ and find an basis for $\text{col}(A)^\perp$.
      What’s the dimension of $\text{col}(A)^\perp$?
   (d) Use Gram-Schmidt process to find an orthogonal basis for the column space of $A$.
   (e) Find an orthonormal basis for the column of the matrix $A$.
   (f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of $A$ and write $y = \hat{y} + z$ where $\hat{y} \in \text{col}(A)$ and $z \in \text{col}(A)^\perp$.
      Also find the shortest distance from $y$ to $\text{Col}(A)$.
   (g) Using previous result to explain why $Ax = y$ has no solution.
   (h) Use orthogonal projection to find the least square solution of $Ax = y$.
   (i) Use normal equation to find the least square solution of $Ax = y$.

5. Find the equation $y = a + mx$ of the least square line that best fits the given data points. $(0, 1), (1, 1), (3, 2)$. 


6. (a) Show that the set of vectors

\[ B = \left\{ u_1 = \left( -\frac{3}{5}, \frac{4}{5}, 0 \right), \ u_2 = \left( \frac{4}{5}, \frac{3}{5}, 0 \right), \ u_3 = (0, 0, 1) \right\} \]

is an orthonormal basis of \( \mathbb{R}^3 \).

(b) Find the coordinates of the vector \((1, -1, 2)\) with respect to the basis in (a).

7. (a) Let \( A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \). Find the inverse matrix of \( A \) if possible.

(b) Find the coordinates of the vector \((1, -1, 2)\) with respect to the basis \( B \) obtained from the column vectors of \( A \).

8. Let \( H = \left\{ \begin{bmatrix} a + 2b - c \\ a - b - 4c \\ a + b - 2c \end{bmatrix} : a, b, \text{any real numbers} \right\} \).

a. Explain why \( H \) is a a subspace of \( \mathbb{R}^3 \).

b. Find a set of vectors that spans \( H \).

c. Find a basis for \( H \).

d. What is the dimension of the subspace?

e. Find an orthogonal basis for \( H \).

9. Determine if the following systems are consistent and if so give all solutions in parametric vector form.

(a)

\[
\begin{align*}
  x_1 - 2x_2 &= 3 \\
  2x_1 - 7x_2 &= 0 \\
 -5x_1 + 8x_2 &= 5
\end{align*}
\]

(b)

\[
\begin{align*}
  x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\
 -x_1 - 2x_2 + 4x_3 - x_4 &= 6 \\
 -2x_1 - 4x_2 + 7x_3 - x_4 &= 1
\end{align*}
\]
10. Let \( A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix} \) which is row reduced to \( \begin{bmatrix} 1 & -3 & -2 & 20 & -3 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) Find a basis for the column space of \( A \)
(b) Find a basis for the nullspace of \( A \)
(c) Find the rank of the matrix \( A \)
(d) Find the dimension of the nullspace of \( A \).
(e) Is \( \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix} \) in the range of \( A \)?
(e) Does \( Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} \) have any solution? Find a solution if it’s solvable.

11. Determine if the columns of the matrix form a linearly independent set. Justify your answer.
\[
\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.
\]

12. Let \( A \) be a \( 12 \times 5 \) matrix. You may assume that \( \text{Nul}(A^T A) = \text{Nul}(A) \).
   (This relation holds form any matrix \( A \).)
   a. What is the size of \( A^T A \)?
   b. Use the Rank Theorem to obtain an equation involving \( \text{rank}(A) \).
   Find another equation involving \( \text{rank}(A^T A) \). What is the connection between these two ranks?
   c. Suppose the columns of \( A \) are linearly independent. Explain why \( A^T A \) is invertible.