

## Solution to Linear Algebra (Math 2890) Review Problems II

1. Find the inverses of the following matrices if they exist.

$$A = \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

Solution: (a) Since  $\det(A) = -1$ , we have  $A^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & -7 \end{bmatrix}$

(b)

$$\begin{aligned} & \left[ \begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \end{array} \right] \\ & r_2 := r_2 + (-1)r_1, r_3 := r_3 + (-1)r_1 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & -1 \end{array} \right] \\ & r_3 := r_3 + r_2 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & 1 & -2 \end{array} \right] \\ & r_3 := \frac{r_3}{2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right] \\ & r_2 := r_2 + (-1)r_3 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right] \end{aligned}$$

So  $B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$ .

(c)

$$\begin{aligned}
& \left[ \begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 5 & 6 & 7 & 0 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r_2 := r_2 - 2r_1} \left[ \begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array} \right] \\
& \xrightarrow{r_2 \leftrightarrow r_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{array} \right] \\
& \xrightarrow{r_2 := r_2 - 2r_1, r_3 := r_3 - 8r_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 9 & 18 & 16 & -8 & 1 \end{array} \right] \\
& \xrightarrow{r_3 := r_3 + (-3)r_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 3 & -1 & 0 \\ 0 & 0 & 0 & 7 & -5 & 1 \end{array} \right]
\end{aligned}$$

So  $C$  only has two pivot vectors and  $C$  is not invertible.

2. Describe the values of  $t$  so that the following matrices are invertible

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & t+1 & 3 \\ 1 & t & t+1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 0 & t \\ -1 & 0 & t & 0 \\ 0 & -t & 0 & 1 \\ -t & 0 & -1 & 0 \end{bmatrix}$$

Solution:

(a)

$$\begin{aligned}
M &= \begin{bmatrix} 1 & 1 & 2 \\ 1 & t+1 & 3 \\ 1 & t & t+1 \end{bmatrix} \xrightarrow{r_2 := r_2 + (-1)r_1, r_3 := r_3 + (-1)r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & t & 1 \\ 0 & t-1 & t-1 \end{bmatrix} \\
& \xrightarrow{r_3 := \frac{1}{t-1}r_3 \text{ if } t-1 \neq 0, r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & t & 1 \end{bmatrix} \xrightarrow{r_3 := r_3 + (-t)r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1-t \end{bmatrix}
\end{aligned}$$

Thus  $M$  is invertible if  $t \neq 1$ .

(b)

$$\begin{aligned}
 A = \begin{bmatrix} 0 & 1 & 0 & t \\ -1 & 0 & t & 0 \\ 0 & -t & 0 & 1 \\ -t & 0 & -1 & 0 \end{bmatrix} & \xrightarrow{\text{interchange 1st and 2nd row, } (-1)r_1} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & -t & 0 & 1 \\ -t & 0 & -1 & 0 \end{bmatrix} \\
 & \xrightarrow{r_4 + t \cdot r_1} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & -t & 0 & 1 \\ 0 & 0 & -1 - t^2 & 0 \end{bmatrix} \xrightarrow{r_3 + t \cdot r_2} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 + t^2 \\ 0 & 0 & -1 - t^2 & 0 \end{bmatrix} \\
 & \xrightarrow{\frac{1}{1+t^2}r_3, \frac{-1}{1+t^2}r_4, r_3 \leftrightarrow r_4} \begin{bmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Thus  $A$  has four pivot vectors and  $A$  is invertible for all  $t$ . Note that we have used the fact that  $1 + t^2 \neq 0$  in the computation.

3. (a) Show that the matrix  $A = \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$  is invertible and find its inverse.

(b) Use previous result to find the inverse of  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ .

Solution: (a) Let  $C = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ . Then  $C$  is the inverse of  $A$  if

$$AC = I. \text{ So we have } \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\iff \begin{bmatrix} X + BY & Y \\ Z + BW & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \iff \begin{bmatrix} X + BY & Y \\ Z + BW & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\iff X + BY = I, Y = 0, Z + BW = 0$$

and  $W = I$

$$\iff Y = 0, W = I, X = I - BY = I - B \cdot 0 = I$$

and  $Z = -BW = -B \cdot I = -B$ .

$$\text{Hence } A^{-1} = \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}.$$

(b) From part (a), we have the inverse of  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$  is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}.$$

4. Find all values of  $a$  and  $b$  so that the subspace of  $\mathbb{R}^4$  spanned by

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ -a \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$  is two-dimensional.

Solution: Consider the matrix  $A = \begin{bmatrix} 0 & b & -2 \\ 1 & 1 & 2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

*interchange first row and second row*  $\begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$r_4 := r_1 + r_4$   $\begin{bmatrix} 1 & 1 & 2 \\ 0 & b & -2 \\ 0 & -a & 0 \\ 0 & 2 & 2 \end{bmatrix}$

*interchange second row and fourth row*  $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$

*divide second row by 2*  $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -a & 0 \\ 0 & b & -2 \end{bmatrix}$   $r_3 := r_3 + ar_2, r_4 := r_4 - br_2$   $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & a \\ 0 & 0 & -2-b \end{bmatrix}$ .

Now the first and second vectors are pivot vectors. So  $\text{rank}(A) = 2$  if  $a = 0$  and  $-2 - b = 0$ .

So  $a = 0$  and  $b = -2$

5. Let  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ . You can assume that  $\mathcal{B}$  is a basis for  $R^3$

(a) Which vector  $x$  has the coordinate vector  $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ .

Let  $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ . So  $x = A[x]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - 3 + 0 \\ 0 - 2 + 0 \\ 0 - 1 + 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$

(b) Find the  $\beta$ -coordinate vector of  $y = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ .

Solution. We have to solve  $Ax = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ .

$$\begin{bmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 2 & 0 & | & -2 \\ 0 & 1 & 2 & | & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2} \begin{bmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 1 & 2 & | & 3 \end{bmatrix} \xrightarrow{-r_2 + r_3} \begin{bmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 2 & | & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_3} \begin{bmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \xrightarrow{-3r_2 + r_1} \begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}.$$

So  $[y]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ .

6. Let

$$M = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

Find bases for  $Col(M)$  and  $Nul(M)$ , and then state the dimensions of these subspaces

Solution:  $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \xrightarrow{r_2 := -r_1 + r_2, r_3 := -r_1 + r_3} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}$

$$r_3 := \widetilde{-2r_2} + r_3 \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_1 := \widetilde{-2r_2} + r_3 \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the first two vectors are pivot vectors and  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$  is a basis

for  $Col(A)$  and  $dim(Col(A)) = 2$ .

The solution to  $Mx = 0$  is  $x_1 + x_3 - x_4 = 0$  and  $x_2 + 2x_3 + x_4 = 0$ . So

$$x = \begin{bmatrix} -x_3 + x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}. \text{ Hence the basis for } Nul(M)$$

is  $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$  and  $dim(Nul(M)) = 2$ .

7. Determine which sets in the following are bases for  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . Justify your answer

(a)  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ . Solution: Since  $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , the set  $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$  is dependent. It is not a basis.

(b)  $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ . Yes. This set forms a basis since they are independent and span  $\mathbb{R}^3$ .

(c)  $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

This is not a basis since it doesn't span  $\mathbb{R}^3$ .

(d)  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . This set forms a basis since they are independent and span  $\mathbb{R}^2$ .

(e)  $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ . This is not a basis since it is de-

pendent.

8. Diagonalize the following matrices if possible.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.

a. 1.  $A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 1 \\ 1 & 1 & 2 - \lambda \end{bmatrix}.$

So  $\det(A - \lambda I) = (2 - \lambda)^3 + 1 + 1 - (2 - \lambda) - (2 - \lambda) - (2 - \lambda) = (4 - 4\lambda + \lambda^2)(2 - \lambda) + 2 - 6 + 3\lambda = 8 - 8\lambda + 2\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 4 + 3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda - 1)^2(\lambda - 4)$ . So the characteristic equation is  $-(\lambda - 1)^2(\lambda - 4) = 0$ .

2. Solving the characteristic equation  $-(\lambda - 1)^2(\lambda - 4) = 0$ , we get that the eigenvalues are  $\lambda = 1$  and  $\lambda = 4$ .

3. When  $\lambda = 1$ , we have

$$A - \lambda I = \begin{bmatrix} 2 - 1 & 1 & 1 \\ 1 & 2 - 1 & 1 \\ 1 & 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{-r_1 + r_2, -r_1 + r_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of  $(A - I)x = 0$  is  $x_1 + x_2 + x_3 = 0$  and  $x_1 = -x_2 - x_3$  So

$$\text{Null}(A - I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The basis for the eigenspace corresponding to eigenvalue 1 is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. When  $\lambda = 4$ , we have

$$A - \lambda I = \begin{bmatrix} 2 - 4 & 1 & 1 \\ 1 & 2 - 4 & 1 \\ 1 & 1 & 2 - 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

*interchange 1st row and 2nd row*  $= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\begin{aligned}
2r_1 + \widetilde{r_2}, -r_1 + r_3 &= \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} r_2/3, r_2 + r_3 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
\widetilde{2r_2 + r_1} &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

The solution of  $(A - 4I)x = 0$  is  $x_1 - x_3 = 0$  and  $x_2 - x_3 = 0$  So

$$\text{Null}(A - I) = \left\{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The basis for the eigenspace corresponding to eigenvalue 4 is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

So  $A$  is diagonalizable with  $A = PDP^{-1}$  where  $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

b.

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$1. B - \lambda I = \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 0 & 2 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{bmatrix}.$$

So  $\det(B - \lambda I) = (2 - \lambda)^2(1 - \lambda)$  and the eigenvalues of  $B$  are  $\lambda = 2$  and  $\lambda = 1$ .

2. When  $\lambda = 2$ , we have

$$B - \lambda I = \begin{bmatrix} 2 - 2 & 1 & 1 \\ 0 & 2 - 2 & 1 \\ 0 & 0 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$r_2 := r_2 + \widetilde{r_3}, r_1 := r_1 + r_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of  $(B - 2I)x = 0$  is  $x_2 = 0$  and  $x_3 = 0$  So  $Null(B - 2I) = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

The basis for the eigenspace corresponding to eigenvalue 2 is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

3. When  $\lambda = 1$ , we have

$$B - \lambda I = \begin{bmatrix} 2-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_1 := \widetilde{r_1} - r_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution of  $(B - I)x = 0$  is  $x_1 = 0$  and  $x_2 + x_3 = 0$  So  $Null(B - I) = \left\{ \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ .

The basis for the eigenspace corresponding to eigenvalue 1 is  $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ .

So  $B$  has only two independent eigenvectors and  $B$  is not diagonalizable.

9. Let  $A$  be the matrix

$$A = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

- Find the eigenvalues and a basis of eigenvectors for  $A$ .
- Diagonalize the matrix  $A$  if possible.
- Find the matrix exponential  $e^A$ .

Solution.

$$1. A - \lambda I = \begin{bmatrix} -3 - \lambda & -4 \\ -4 & 3 - \lambda \end{bmatrix}.$$

$$\text{So } \det(A - \lambda I) = (-3 - \lambda)(3 - \lambda) - 16 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$$

$$\text{So the characteristic equation is } (\lambda - 5)(\lambda + 5) = 0.$$

2. Solving the characteristic equation  $(\lambda - 5)(\lambda + 5) = 0$ , we get that the eigenvalues are  $\lambda = 5$  and  $\lambda = -5$ .

3. When  $\lambda = 5$ , we have

$$A - \lambda I = \begin{bmatrix} -3 - 5 & -4 \\ -4 & 3 - 5 \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -4 & 2 \end{bmatrix} \xrightarrow{-r_1/2 + r_2} \begin{bmatrix} -8 & -4 \\ 0 & 0 \end{bmatrix}.$$

The solution of  $(A - 5I)x = 0$  is  $-8x_1 - 4x_2 = 0$ , i.e. and  $x_2 = -2x_1$ . So  $\text{Null}(A - I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ .

The basis for the eigenspace corresponding to eigenvalue 5 is  $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

4. When  $\lambda = -5$ , we have

$$A - \lambda I = \begin{bmatrix} -3 + 5 & -4 \\ -4 & 3 + 5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \xrightarrow{2r_1 + r_2} \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}.$$

The solution of  $(A + 5I)x = 0$  is  $2x_1 - 4x_2 = 0$ , i.e. and  $x_1 = 2x_2$ . So  $\text{Null}(A - I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ .

The basis for the eigenspace corresponding to eigenvalue  $-5$  is  $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$$\text{Let } P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}. \text{ Then } P^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

$$\text{So } A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{aligned} \text{and } e^A &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^5 & 0 \\ 0 & e^{-5} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5}e^5 & -\frac{2}{5}e^5 \\ \frac{2}{5}e^{-5} & \frac{1}{5}e^{-5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5}e^5 + \frac{4}{5}e^{-5} & -\frac{2}{5}e^5 + \frac{2}{5}e^{-5} \\ -\frac{2}{5}e^5 + \frac{2}{5}e^{-5} & \frac{2}{5}e^5 + \frac{1}{5}e^{-5} \end{bmatrix}. \end{aligned}$$

10. Let  $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$ .

(a) What is the column space of  $A$ ?

(b) Describe the subspace  $\text{col}(A)^\perp$  and find an basis for  $\text{col}(A)^\perp$ .

(c) Use Gram-Schmidt process to find an orthogonal basis for the column of the matrix  $A$ .

(d) Find an orthonormal basis for the column of the matrix  $A$ .

(e) Find the orthogonal projection of  $y = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix}$  onto the column

space of  $A$  and write  $y = \hat{y} + z$  where  $\hat{y} \in \text{col}(A)$  and  $z \in \text{col}(A)^\perp$ . Also find the shortest distance from  $y$  to  $\text{Col}(A)$ .

Solution: (a) The column space is the subspace spanned by the

column vectors. So  $\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right\}$ .

(b)  $\text{col}(A)^\perp = \{x \mid x \cdot y = 0 \text{ for all } y \in \text{col}(A)\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid -x_1 + 3x_2 + x_3 + x_4 = 0, 6x_1 - 8x_2 - 2x_3 - 4x_4 = 0, 6x_1 + 3x_2 + 6x_3 - 3x_4 = 0 \right\}$$

$$\text{Consider } \left[ \begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 6 & -8 & -2 & -4 & 0 \\ 6 & 3 & 6 & -3 & 0 \end{array} \right] \xrightarrow{6r_1 + r_2, 6r_1 + r_3} \left[ \begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 0 & 10 & 4 & 2 & 0 \\ 0 & 21 & 12 & 3 & 0 \end{array} \right]$$

$$\xrightarrow{-\frac{21}{10}r_2 + r_3} \left[ \begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 0 & 10 & 4 & 2 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \end{array} \right]$$

$$\xrightarrow{-\frac{20}{18}r_3 + r_2, -\frac{5}{18}r_3 + r_1} \left[ \begin{array}{cccc|c} -1 & 3 & 0 & 4/3 & 0 \\ 0 & 10 & 0 & 10/3 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \end{array} \right]$$

$$\begin{array}{l} \widetilde{-\frac{21}{10}r_2 + r_3} \\ \widetilde{-\frac{3}{10}r_2 + r_1} \\ \widetilde{-r_1, \frac{1}{10}r_2, \frac{5}{18}r_3} \end{array} \left[ \begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 0 & 10 & 4 & 2 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \\ -1 & 0 & 0 & 1/3 & 0 \\ 0 & 10 & 0 & 10/3 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \\ 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{array} \right]$$

So  $x_1 - \frac{1}{3}x_4 = 0$ ,  $x_2 + \frac{1}{3}x_4 = 0$  and  $x_3 - \frac{1}{3}x_4 = 0$ . This implies

that  $x_1 = \frac{1}{3}x_4$ ,  $x_2 = -\frac{1}{3}x_4$ ,  $x_3 = \frac{1}{3}x_4$  and  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_4 \\ -\frac{1}{3}x_4 \\ \frac{1}{3}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$ . Hence  $\text{col}(A)^\perp = \text{span}\left\{ \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$  and  $\left\{ \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$  is a basis for  $\text{col}(A)^\perp$ .

Let  $w_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ ,  $w_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$  and  $w_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$ .

Gram-Schmidt process is

$$v_1 = w_1, v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1 \text{ and } v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2.$$

So  $v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ . Compute  $w_2 \cdot v_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = -36$ ,  $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 12$  and  $v_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{(-36)}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ .

Compute  $w_3 \cdot v_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 6$ ,  $w_3 \cdot v_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 30$ ,

$v_2 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 12$  and

$$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

Hence  $\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$  is an orthogonal basis for  $Col(A)$ .

$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} = \left\{ \begin{bmatrix} -\frac{1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix} \right\}$  is an orthonormal basis for  $Col(A)$ .

$$(e) \ y = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix}.$$

Since  $\left\{ v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$  is an orthogonal basis for  $Col(A)$ ,  $y = \hat{y} + z$  where  $\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \in$

$Col(A)$  and  $z = y - \hat{y} \in Col(A)^\perp$ . Compute  $y \cdot v_1 = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} =$

$$1 + 24 - 6 + 4 = 23, \quad v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 1 + 9 + 1 + 1 = 12,$$

$$y \cdot v_2 = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = -3 + 8 - 6 - 4 = -5, \quad v_2 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 9 + 1 + 1 + 1 = 12,$$

$$y \cdot v_3 = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = 1 - 8 - 18 - 4 = -29, \quad v_3 \cdot v_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = 1 + 1 + 9 + 1 = 12.$$

$$\text{So } \hat{y} = \frac{23}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} + \frac{(-5)}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{(-29)}{12} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 31/4 \\ -23/4 \\ 19/4 \end{bmatrix} \text{ and } z =$$

$$y - \hat{y} = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} - \begin{bmatrix} -3/4 \\ 31/4 \\ -23/4 \\ 19/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ -3/4 \end{bmatrix}.$$

The shortest distance from  $y$  to  $Col(A) = \|y - \hat{y}\| = \|z\| = \sqrt{(-1/4)^2 + (1/4)^2 + (-1/4)^2 + (-3/4)^2} = \sqrt{12/16} = \sqrt{3/4}$