

## Linear Algebra (Math 2890) Solution to Final Review Problems

1. Let  $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$ .

- (a) Find the condition on  $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$  such that  $Ax = b$  is solvable.

Solution:

Consider the augmented matrix  $[A \ b] = \begin{bmatrix} 1 & 2 & 2 & | & b_1 \\ 1 & 1 & 0 & | & b_2 \\ 0 & 1 & 2 & | & b_3 \\ -1 & 0 & -1 & | & b_4 \end{bmatrix}$

$$a_2 := \widetilde{a_2} + (-1)a_1 \begin{bmatrix} 1 & 2 & 2 & | & b_1 \\ 0 & -1 & -2 & | & b_2 - b_1 \\ 0 & 1 & 2 & | & b_3 \\ -1 & 0 & -1 & | & b_4 \end{bmatrix}$$

$$a_4 := \widetilde{a_4} + a_1 \begin{bmatrix} 1 & 2 & 2 & | & b_1 \\ 0 & -1 & -2 & | & b_2 - b_1 \\ 0 & 1 & 2 & | & b_3 \\ 0 & 2 & 1 & | & b_4 + b_1 \end{bmatrix}$$

$$a_2 := \widetilde{a_2} - a_2 \begin{bmatrix} 1 & 2 & 2 & | & b_1 \\ 0 & 1 & 2 & | & -b_2 + b_1 \\ 0 & 1 & 2 & | & b_3 \\ 0 & 2 & 1 & | & b_4 + b_1 \end{bmatrix}$$

$$a_3 := a_3 - \widetilde{a_2}, a_4 := a_4 - 2a_2 \left[ \begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \\ 0 & 0 & -3 & b_4 - b_1 + 2b_2 \end{array} \right]$$

$$\widetilde{a_3} \leftrightarrow a_4 \left[ \begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 2 & -b_2 + b_1 \\ 0 & 0 & -3 & b_4 - b_1 + 2b_2 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{array} \right]$$

From here, we can see that  $Ax = b$  has a solution if  $b_3 + b_2 - b_1 = 0$ .

(b) What is the column space of  $A$ ?

Solution:

The column space is the subspace spanned by the column vectors.

$$\text{So } \text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

(c) Describe the subspace  $\text{col}(A)^\perp$  and find an basis for  $\text{col}(A)^\perp$ .

Solution:  $\text{col}(A)^\perp = \{x \mid x \cdot y = 0 \text{ for all } y \in \text{col}(A)\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_1 + x_2 - x_4 = 0, 2x_1 + x_2 + x_3 = 0, 2x_1 + 2x_3 - x_4 = 0 \right\}$$

$$\text{Consider } \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix} \xrightarrow{r_2 := r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{bmatrix}$$

$$\begin{array}{l}
\widetilde{r_3 := r_3 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & 2 & 1 \end{bmatrix} \\
\widetilde{r_3 := r_3 + 2r_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\
\widetilde{r_3 := r_3 / (-3)} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
\quad
\begin{array}{l}
\widetilde{r_2 := -r_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 2 & 1 \end{bmatrix} \\
\widetilde{r_1 := r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\
\widetilde{r_1 := r_1 - r_3, r_2 := r_2 + 2r_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}$$

So  $x_1 + x_3 = 0$ ,  $x_2 - x_3 = 0$  and  $x_4 = 0$ ,  $x_3$  is free. This implies that  $x_1 = -x_3$ ,  $x_2 = x_3$ ,  $x_4 = 0$  and  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ .

Hence  $col(A)^\perp = span\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$  and  $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$  is a basis for  $col(A)^\perp$ .

The dimension of  $col(A)^\perp$  is 1.

- (d) Use Gram-Schmidt process to find an orthogonal basis for the column of the matrix  $A$ .

Solution:

$$\text{Let } w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } w_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}.$$

Gram-Schmidt process is

$$v_1 = w_1, v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1 \text{ and } v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2.$$

$$\text{So } v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \text{ Compute } w_2 \cdot v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3, v_1 \cdot v_1 =$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3 \text{ and } v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{Compute } w_3 \cdot v_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3, w_3 \cdot v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3,$$

$$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3 \text{ and}$$

$$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} 2-1-1 \\ 0-1-0 \\ 2-0-1 \\ -1+1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}. \text{ Hence } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\} \text{ is an orthog-} \\ \text{onal basis for } \text{Col}(A).$$

- (e) Find an orthonormal basis for the column of the matrix  $A$ .

Solution:

Note that  $\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{3}$ ,  $\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{3}$  and

$$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{3}. \text{ Hence } \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

is an orthonormal basis for  $\text{Col}(A)$ .

- (f) Find the orthogonal projection of  $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$  onto the column

space of  $A$  and write  $y = \hat{y} + z$  where  $\hat{y} \in \text{col}(A)$  and  $z \in \text{col}(A)^\perp$ . Also find the shortest distance from  $y$  to  $\text{Col}(A)$ .

Solution: Since  $\{v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}\}$  is an

orthogonal basis for  $\text{Col}(A)$ ,  $y = \hat{y} + z$  where  $\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \in \text{Col}(A)$  and  $z = y - \hat{y} \in \text{Col}(A)^\perp$ . Compute

$$y \cdot v_1 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 7 + 3 + 0 + 2 = 12, v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \\ 1 + 1 + 1 = 3, y \cdot v_2 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 7 + 0 + 10 - 2 = 15,$$

$$v_2 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3,$$

$$y \cdot v_3 = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0 - 3 + 10 + 2 = 9, \quad v_3 \cdot v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3.$$

$$\text{So } \hat{y} = \frac{12}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{(15)}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{9}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4+5+0 \\ 4+0-3 \\ 0+5+3 \\ -4+5-3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix} \text{ and}$$

$$z = y - \hat{y} = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix} - \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}. \text{ Note that } z \in \text{Col}(A)^\perp = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

The shortest distance from  $y$  to  $\text{Col}(A) = \|y - \hat{y}\| = \|z\| = \sqrt{(2)^2 + (-2)^2 + (2)^2 + (0)^2} = \sqrt{12}$ .

- (g) Using previous result to explain why  $Ax = y$  has no solution.  
 Solution: Since the orthogonal projection of  $y$  to  $\text{Col}(A)$  is not  $y$ , this implies that  $y$  is not in  $\text{Col}(A)$ . So  $Ax = y$  has no solution.
- (h) Use orthogonal projection to find the least square solution of  $Ax = y$ .

Solution: The least square solution of  $Ax = y$  is the solution of  $Ax = \hat{y} = \begin{bmatrix} 9 \\ 1 \\ 8 \\ -2 \end{bmatrix}$  where  $\hat{y}$  is the orthogonal projection of  $y$  onto the column space of  $A$ .

Consider the augmented matrix

$$[A \hat{y}] = \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 8 \\ -1 & 0 & -1 & -2 \end{array} \right] \quad r_2 := r_2 - r_1, r_3 := r_3 + r_1 \quad \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 1 & 7 \end{array} \right]$$

$$r_3 := r_3 + r_2, r_4 := r_4 + r_1 \quad \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -9 \end{array} \right]$$

$$r_2 := -r_2, r_4 := \widetilde{r_4}/(-3), r_3 \leftrightarrow r_4 \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$r_2 := r_2 - 2r_3, r_1 := r_1 - 2r_3 \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$r_1 := \widetilde{r_1} - 2r_2 \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

So  $x_1 = -1$ ,  $x_2 = 2$ ,  $x_3 = 3$  and the least square solution of

$$Ax = y \text{ is } x = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

2. (a) Show that the set of vectors

$$B = \left\{ u_1 = \left( -\frac{3}{5}, \frac{4}{5}, 0 \right), u_2 = \left( \frac{4}{5}, \frac{3}{5}, 0 \right), u_3 = (0, 0, 1) \right\}$$

is an **orthonormal basis** of  $\mathbb{R}^3$ .

Solution: Compute  $u_1 \cdot u_2 = \left( -\frac{3}{5}, \frac{4}{5}, 0 \right) \cdot \left( \frac{4}{5}, \frac{3}{5}, 0 \right) = \frac{-12}{5} + \frac{12}{5} = 0$ ,  
 $u_1 \cdot u_3 = \left( -\frac{3}{5}, \frac{4}{5}, 0 \right) \cdot (0, 0, 1) = 0$ ,  $u_2 \cdot u_3 = \left( \frac{4}{5}, \frac{3}{5}, 0 \right) \cdot (0, 0, 1) = 0$ ,  
 $u_1 \cdot u_1 = \left( -\frac{3}{5}, \frac{4}{5}, 0 \right) \cdot \left( -\frac{3}{5}, \frac{4}{5}, 0 \right) = \frac{9}{25} + \frac{16}{25} = 1$ ,  $u_3 \cdot u_3 = (0, 0, 1) \cdot (0, 0, 1) = 1$ ,  
 $u_2 \cdot u_2 = \left( \frac{4}{5}, \frac{3}{5}, 0 \right) \cdot \left( \frac{4}{5}, \frac{3}{5}, 0 \right) = \frac{16}{25} + \frac{9}{25} = 1$

(b) Find the coordinates of the vector  $(1, -1, 2)$  with respect to the basis in (a).

Solution: Let  $y = (1, -1, 2)$ . So  $y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3 = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + (y \cdot u_3)u_3$ . Compute  $y \cdot u_1 = (1, -1, 2) \cdot \left( -\frac{3}{5}, \frac{4}{5}, 0 \right) =$

$$-\frac{3}{5} - \frac{4}{5} = -\frac{7}{5}, y \cdot u_2 = (1, -1, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}, y \cdot u_3 = (1, -1, 2) \cdot (0, 0, 1) = 2.$$

So the coordinate of  $y$  with respect to the basis in (a) is  $(-\frac{7}{5}, \frac{1}{5}, 2)$ .

3. (a) Let  $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ . Find the inverse matrix of  $A$  if possible.

Solution: Consider the augmented matrix  $[A \ I] = \left[ \begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$

$$r_1 := \widetilde{r_1 - r_3} \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$r_3 := \widetilde{r_3 - 2r_1} \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array} \right]$$

$$r_2 := \widetilde{r_2 + r_3} \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & -1 & -1 & -2 & 1 & 3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array} \right] \quad r_2 := \widetilde{-r_2} \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & -3 & -2 & -2 & 0 & 3 \end{array} \right]$$

$$r_3 := \widetilde{r_3 + 3r_2} \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right]$$

$$r_2 := \widetilde{r_2 - r_3}, r_1 := \widetilde{r_1 - 3r_3} \left[ \begin{array}{ccc|ccc} 1 & 3 & 0 & -11 & 9 & 17 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right]$$

$$r_1 := \widetilde{r_1 - 3r_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right].$$

So  $A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$ .

(b) Find the coordinates of the vector  $(1, -1, 2)$  with respect to the basis  $B$  obtained from the column vectors of  $A$ .

Solution: The coordinate is  $x = A^{-1} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -5 \end{bmatrix}$ .

4. Let  $A$  be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) Prove that  $\det(A - \lambda I) = (1 - \lambda)^2(4 - \lambda)$ .

Solution: Compute  $A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 1 \\ 1 & 1 & 2 - \lambda \end{bmatrix}$  and

$$\det(A - \lambda I) = (2 - \lambda)^3 + 1 + 1 - (2 - \lambda) - (2 - \lambda) - (2 - \lambda) = 8 - 12\lambda + 6\lambda^2 - \lambda^3 + 2 - 6 + 3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (1 - \lambda)^2(4 - \lambda).$$

(b) Orthogonally diagonalizes the matrix  $A$ , giving an orthogonal matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^t$

Solution: We know that the eigenvalues are 1, 1 and 4.

When  $\lambda = 1$ ,  $A - (1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x \in \text{Null}(A - I)$  if  $x_1 + x_2 + x_3 = 0$ . So  $x_1 = -x_2 - x_3$  and

$$x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \text{ Thus } \{w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, w_2 =$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$  is a basis for  $\text{Null}(A - (-1)I)$ .

Now we use Gram-Schmidt process to find an orthogonal basis for  $\text{Null}(A - I)$ .

Let  $v_1 = w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$  and  $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$ . Compute  $w_2 \cdot v_1 =$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1 \text{ and } v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2.$$

$$\text{So } v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{2}\right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Hence  $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\}$  is an orthogonal basis for  $\text{Null}(A - I)$ .

When  $\lambda = 4$ ,  $A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$  *interchange  $r_1$  and  $r_2$ ,*

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$-2r_1 + r_2, -r_1 + r_3 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$r_2 + r_3, r_2/(-3) \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad 2r_2 + r_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad x \in \text{Null}(A -$$

$4I)$  if  $x_1 - x_3 = 0$  and  $x_2 - x_3 = 0$ . So  $x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . Thus

$\{v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$  is a basis for  $\text{Null}(A - 4I)$ .

So  $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$  is an orthogonal basis for  $R^3$  which are eigenvectors corresponding to  $\lambda = 1$ ,  $\lambda = 1$  and  $\lambda = 4$ . Compute  $\|v_1\| = \sqrt{2}$ ,  $\|v_2\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$  and  $\|v_3\| = \sqrt{3}$ .

Thus  $\{\frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \frac{v_2}{\|v_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \frac{v_3}{\|v_3\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\}$  is an orthonormal basis for  $R^3$  which are eigenvectors corresponding to  $\lambda = 1$ ,  $\lambda = 1$  and  $\lambda = 4$ .

Finally, we have  $A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} P^T$  where  $P = [\frac{v_1}{\|v_1\|} \ \frac{v_2}{\|v_2\|} \ \frac{v_3}{\|v_3\|}] =$

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(c) Find  $A^{10}$  and  $e^A$ .

So  $A^{10} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{10} \end{bmatrix} P^T$  and  $e^A = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^T$

5. Classify the quadratic forms for the following quadratic forms. Make a change of variable  $x = Py$ , that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

(a)  $9x_1^2 - 8x_1x_2 + 3x_2^2$ .

Let  $Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2 = x^T \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} x$  and  $A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$ . We want to orthogonally diagonalize  $A$ .

Compute  $A - \lambda I = \begin{bmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$  and  $\det(A - \lambda I) = (9 - \lambda)(3 - \lambda) - 16 = \lambda^2 - 12\lambda + 27 - 16 = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11)$ . So  $\lambda = 1$  or  $\lambda = 11$ . Since the eigenvalues of  $A$  are all positive, we know that the quadratic form is positive definite.

Now we diagonalize  $A$ .

$\lambda = 1$ :  $A - 1 \cdot I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ . So  $x \in \text{Null}(A - 1 \cdot I)$  iff  $2x_1 - x_2 = 0$ . So  $x_2 = 2x_1$  and  $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . So  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is an eigenvector corresponding to eigenvalue  $\lambda = 1$ .

$\lambda = 11$ :  $A - 11 \cdot I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ . So  $x \in \text{Null}(A - 11 \cdot I)$  iff  $x_1 + 2x_2 = 0$ . So  $x_1 = -2x_2$  and  $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ . So  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  is an eigenvector corresponding to eigenvalue  $\lambda = 11$ .

Now  $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$  is an orthogonal basis. Compute  $\|v_1\| = \sqrt{5}$  and  $\|v_2\| = \sqrt{5}$ . Thus  $\left\{ \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\}$  is an orthonormal basis of eigenvectors. So we have  $A = P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T$  where  $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ .

Now  $Q(x) = x^T Ax = x^T P \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} P^T x = y^T \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} y = y_1^2 + 11y_2^2$  if  $y = P^T x$ . So  $Py = PP^T x$ ,  $x = Py$  and  $P \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ . Note that we have used the fact that  $PP^T = I$ .

(b)  $-5x_1^2 + 4x_1x_2 - 2x_2^2$ .

Let  $Q(x_1, x_2) = -5x_1^2 + 4x_1x_2 - 2x_2^2 = x^T \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} x$  and  $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ . We want to orthogonally diagonalize  $A$ .

Compute  $A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$  and  $\det(A - \lambda I) = (-5 - \lambda)(-2 - \lambda) - 4 = \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = (\lambda + 1)(\lambda + 6)$ . So  $\lambda = -1$  or  $\lambda = -6$ . Since the eigenvalues of  $A$  are all negative, we know that the quadratic form is negative definite.

Now we diagonalize  $A$ .

$\lambda = -1$ :  $A - (-1) \cdot I = \begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ . So  $x \in \text{Null}(A - 1 \cdot I)$  iff  $2x_1 - x_2 = 0$ . So  $x_2 = 2x_1$  and  $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . So  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is an eigenvector corresponding to eigenvalue  $\lambda = -1$ .

$\lambda = -6$ :  $A - (-6) \cdot I = \begin{bmatrix} -5-(-6) & 2 \\ 2 & (-2)-(-6) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ . So  $x \in \text{Null}(A - 11 \cdot I)$  iff  $x_1 + 2x_2 = 0$ . So  $x_1 = -2x_2$  and  $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ . So  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  is an eigenvector corresponding to eigenvalue  $\lambda = -6$ .

Now  $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$  is an orthogonal basis. Compute  $\|v_1\| = \sqrt{5}$  and  $\|v_2\| = \sqrt{5}$ . Thus  $\left\{ \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\}$  is an orthonormal basis of eigenvectors. So we have  $A = P \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} P^T$  where  $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ .

Now  $Q(x) = x^T A x = x^T P \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} P^T x = y^T \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} y = -y_1^2 - 6y_2^2$  if  $y = P^T x$ . So  $P y = P P^T x$ ,  $x = P y$  and  $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ .

(c)  $8x_1^2 + 6x_1x_2$ .

Let  $Q(x_1, x_2) = 8x_1^2 + 6x_1x_2 = x^T \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix} x$  and  $A = \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}$ . We want to orthogonally diagonalize  $A$ .

Compute  $A - \lambda I = \begin{bmatrix} 8-\lambda & 3 \\ 3 & 0-\lambda \end{bmatrix}$  and  $\det(A - \lambda I) = (8 - \lambda) \cdot (-\lambda) - 9 = \lambda^2 - 8\lambda - 9 = (\lambda + 1)(\lambda - 9)$ . So  $\lambda = -1$  or  $\lambda = 9$ . Since  $A$  has positive and negative eigenvalues, we know that the quadratic form is indefinite.

Now we diagonalize  $A$ .

$\lambda = -1$ :  $A - (-1) \cdot I = \begin{bmatrix} 8-(-1) & 3 \\ 3 & 0-(-1) \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ . So  $x \in \text{Null}(A - 1 \cdot I)$  iff  $3x_1 + x_2 = 0$ . So  $x_2 = -3x_1$  and  $x = \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ . So  $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$  is an eigenvector corresponding to eigenvalue  $\lambda = -1$ .

$\lambda = 9$ :  $A - 9 \cdot I = \begin{bmatrix} 8-9 & 3 \\ 3 & 0-9 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ . So  $x \in \text{Null}(A - 9 \cdot I)$  iff  $x_1 - 3x_2 = 0$ . So  $x_1 = 3x_2$  and  $x = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . So  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  is an eigenvector corresponding to eigenvalue  $\lambda = 9$ .

Now  $\{v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$  is an orthogonal basis. Compute  $\|v_1\| = \sqrt{10}$  and  $\|v_2\| = \sqrt{10}$ . Thus  $\left\{ \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{bmatrix}, \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} \right\}$  is an orthonormal basis of eigenvectors. So we have  $A = P \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^T$  where  $P = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$ .  
 Now  $Q(x) = x^T A x = x^T P \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} P^T x = y^T \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix} y = -y_1^2 + 9y_2^2$  if  $y = P^T x$ . So  $Py = PP^T x$ ,  $x = Py$  and  $P \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$ .

6. Let  $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$ .

- (a) Find a basis for the column space of  $A$
- (b) Find a basis for the nullspace of  $A$
- (c) Find the rank of the matrix  $A$
- (d) Find the dimension of the nullspace of  $A$ .

(e) Is  $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$  in the range of  $A$ ?

(e) Does  $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$  have any solution? Find a solution if it's solvable.

Solution: Consider the augmented matrix  $\left[ \begin{array}{ccccc|cc} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 2 & -6 & 9 & -1 & 8 & 4 & 3 \\ 2 & -6 & 9 & -1 & 9 & 3 & 2 \\ -1 & 3 & -4 & 2 & -5 & 1 & 0 \end{array} \right]$

$$-2r_1 + r_2, \widetilde{-2r_1 + r_3}, r_1 + r_4 \left[ \begin{array}{ccccc|cc} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array} \right]$$

$$\begin{aligned}
& \widetilde{-r_2 + r_3} \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & | & 1 & | & 0 \\ 0 & 0 & 1 & 3 & -2 & | & 2 & | & 3 \\ 0 & 0 & 0 & 0 & 1 & | & -1 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & | & 2 & | & 0 \end{bmatrix} \\
& 2r_3 + \widetilde{r_2}, -5r_3 + r_1 \begin{bmatrix} 1 & -3 & 4 & -2 & 0 & | & 6 & | & 5 \\ 0 & 0 & 1 & 3 & 0 & | & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & -1 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & | & 2 & | & 0 \end{bmatrix} \\
& \widetilde{-4r_2 + r_1} \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & | & 6 & | & 1 \\ 0 & 0 & 1 & 3 & 0 & | & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & -1 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & | & 2 & | & 0 \end{bmatrix}.
\end{aligned}$$

So the first, third and fifth vector forms a basis for  $\text{Col}(A)$ , i.e  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$

is a basis for  $\text{Col}(A)$ . The rank of  $A$  is 3 and the dimension of the null space is  $5 - 3 = 2$ .

$x \in \text{Null}(A)$  if  $x_1 - 3x_2 - 14x_4 = 0$ ,  $x_3 + 3x_4 = 0$  and  $x_5 = 0$ . So

$$x = \begin{bmatrix} 3x_2 + 14x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}. \text{ Thus } \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis}$$

for  $\text{NULL}(A)$ .

From the result of row reduction, we can see that  $Ax = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$  is incon-

sistent (not solvable) and  $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$  is not in the range of  $A$ .

From the result of row reduction, we can see that  $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$  is solvable.

7. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.$$

Solution:  $\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 - 1 = 1 \neq 0$ . So the columns of the matrix form a linearly independent set.

$\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}$ . The second column vector is a multiple of the first column vector. So the columns of the matrix form a linearly dependent set.

$$\begin{array}{l} \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \\ \begin{array}{l} \widetilde{r_3 + 4r_1, r_4 + (-5)r_1} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \end{array} \\ \begin{array}{l} \widetilde{r_3 + 3r_2, r_4 + (-4)r_2} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \end{array} \end{array} \quad \begin{array}{l} \text{interchange first and third row} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix} \\ \begin{array}{l} \widetilde{(-1)r_2} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \end{array} \\ \begin{array}{l} \widetilde{\text{interchange 3rd and 4th row, } \frac{1}{7}r_4} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \end{array}$$

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in  $R^4$ .

8. (a) What's the geometric meaning of  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ ? Draw a picture

to indicate the subspace  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ .

Solution:  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\} = \left\{t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ . So  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$  is the line thru  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

- (b) What's the geometric meaning of  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ ? Draw a

picture to indicate the subspace  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

Solution:  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\} = \left\{x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mid x_1, x_2 \in R\right\}$ .

So  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$  is the plane spanned by the vectors  $\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ .

9. Circle True or False:

**T** **F** The matrix  $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$  is diagonalizable

**T** Because  $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$  has three distinct eigenvalues.

**T** **F** The matrix  $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$  is orthogonally diagonalizable

**F** Because  $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$  is not symmetric. Recall that a matrix is orthogonally diagonalizable if and only if it's symmetric.

**T** **F** An orthogonal  $n \times n$  matrix times an orthogonal  $n \times n$  matrix is orthogonal

**T** Suppose  $A$  and  $B$  are orthogonal. Then  $AA^T = A^T A = I$ ,

$$BB^T = B^T B = I, (AB) \cdot (AB)^T = AB B^T A^T = A I A^T = AA^T = I.$$

Similarly, we have  $(AB)^T AB = B^T A^T AB = I$ . Note that we have used the fact that  $(AB)^T = B^T A^T$ .

**T** **F** A  $5 \times 5$  orthogonally diagonalizable matrix has an orthonormal set of 5 eigenvectors

**T**  $A$  is orthogonally diagonalizable if  $A = PDP^T$ . Recall that the column vectors of  $P$  are eigenvectors and it is an orthonormal basis.

**T** **F** A square matrix that has the zero eigenvalue is not invertible

**T** A matrix  $A$  has the zero eigenvalue if there exists a nonzero vector  $x$  such that  $Ax = 0x = 0$ . So  $Ax = 0$  has nonzero solution and  $A$  is not invertible.

- T F** A subspace of dimension 3 can not have a spanning set of 4 vectors
- F** Let  $S = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$  Then  $\dim(S) = 3$  and it is spanned by 4 vectors.
- T F** A subspace of dimension 3 can not have a linearly independent set of 4 vectors
- T** A subspace of dimension 3 have at most three linearly independent set of vectors
- T F** The characteristic polynomial of a  $2 \times 2$  matrix is always a polynomial of degree 2
- T** The characteristic polynomial of a  $n \times n$  matrix is always a polynomial of degree  $n$ .
- T F** If the characteristic polynomial of a matrix is  $(\lambda - 4)^3(\lambda - 1)^2$  and the eigenspace associated to  $\lambda = 4$  has dimension 3, than the matrix is diagonalizable
- F** Because the eigenspace associated to  $\lambda = 4$  has dimension 3 and the eigenspace associated to  $\lambda = 1$  could have dimension 1, then we may not have five independent eigenvectors. So the matrix is not necessarily diagonalizable.

- T F** If the characteristic polynomial of a matrix is  $(\lambda - 4)^3(\lambda - 1)\lambda - 2$  and the eigenspace associated to  $\lambda = 4$  has dimension 3, then the matrix is diagonalizable
- T** Because the eigenspace associated to  $\lambda = 4$  has dimension 3, the eigenspace associated to  $\lambda = 1$  have dimension 1 and the eigenspace associated to  $\lambda = 2$  have dimension 1, then we may not have five independent eigenvectors. So the matrix is diagonalizable.
- T F** The columns of an orthogonal matrix are orthonormal vectors
- T** This is true by the definition of an orthogonal matrix.
- T F**  $AB = BA$  for any  $n \times n$  matrices  $A$  and  $B$
- F** The matrix multiplication is not necessarily commutative.
- T F**  $\det(A + B) = \det A + \det B$  for any  $n \times n$  matrices  $A$  and  $B$
- F** This is false. For example,  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . Then  $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  
 $\det(A) = \det(B) = 0$  and  $\det(A + B) = \det(I) = 1$ .
- T F** Any upper triangular matrix is always diagonalizable.
- F** It may not have enough eigenvectors. For example,  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  is upper triangular matrix. But it has only one eigenvector. So it is not diagonalizable.