Linear Algebra (Math 2890) Review Problems for Final Exam

1. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.

   (a) Find the condition on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $Ax = b$ is solvable.

   (b) What is the column space of A?

   (c) Describe the subspace $col(A)^{\perp}$ and find an basis for $col(A)^{\perp}$. What's the dimension of $col(A)^{\perp}$?

   (d) Use Gram-Schmidt process to find an orthogonal basis for the column space of A.

   (e) Find an orthonormal basis for the column of the matrix A.

   (f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of A and write $y = \widehat{y} + z$ where $\widehat{y} \in col(A)$ and $z \in col(A)^{\perp}$. Also find the shortest distance from y to $Col(A)$.

   (g) Using previous result to explain why $Ax = y$ has no solution.

   (h) Use orthogonal projection to find the least square solution of $Ax = y$.

2. (a) Show that the set of vectors

$$B = \left\{ u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right),\ u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right),\ u_3 = (0, 0, 1) \right\}$$

   is an **orthonormal basis** of $\mathbb{R}^3$.

   (b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis in (a).

3. (a) Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Find the inverse matrix of A if possible.

(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis B obtained from the column vectors of A.

4. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) Prove that $det(A - \lambda) = (1 - \lambda)^2(4 - \lambda)$.

(b) Orthogonally diagonalizes the matrix A, giving an orthogonal matrix P and a diagonal matrix D such that $A = PDP^t$

(c) Find A^{10} and e^A.

5. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

(a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.

(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.

(c) $8x_1^2 + 6x_1x_2$.

6. Let $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$.

(a) Find a basis for the column space of A

(b) Find a basis for the nullspace of A

(c) Find the rank of the matrix A

(d) Find the dimension of the nullspace of A.

(e) Is $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ in the range of A?

(e) Does $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ have any solution? Find a solution if it's solvable.

7. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.$$

8. (a) What's the geometric meaning of $span\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\}$? Draw a picture to indicate the subspace $span\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\}$.

(b) What's the geometric meaning of $span\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$? Draw a picture to indicate the subspace $span\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$.

9. Circle True or False:

T **F** The matrix $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is diagonalizable

T **F** The matrix $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is orthogonally diagonalizable

T **F** An orthogonal $n \times n$ matrix times an orthogonal $n \times n$ matrix is orthogonal

T **F** A 5×5 orthogonally diagonalizable matrix has an orthonormal set of 5 eigenvectors

T **F** A square matrix that has the zero eigenvalue is not invertible

T **F** A subspace of dimension 3 can not have a spanning set of 4 vectors

T **F** A subspace of dimension 3 can not have a linearly independent set of 4 vectors

T **F** The characteristic polynomial of a 2×2 matrix is always a polynomial of degree 2

T **F** If the characteristic polynomial of a matrix is $(\lambda - 4)^3(\lambda - 1)^2$ and the eigenspace associated to $\lambda = 4$ has dimension 3, then the matrix is diagonalizable

T **F** If the characteristic polynomial of a matrix is $(\lambda - 4)^3(\lambda - 1)\lambda - 2)$ and the eigenspace associated to $\lambda = 4$ has dimension 3, than the matrix is diagonalizable

T **F** The columns of an orthogonal matrix are orthonormal vectors

T **F** $AB = BA$ for any $n \times n$ matrices A and B

T **F** $\det(A + B) = \det A + \det B$ for any $n \times n$ matrices A and B

T **F** Any upper triangular matrix is always diagonalizable.