Solutions to Linear Algebra Practice Problems 1

1. Determine which of the following augmented matrices are in row echelon from, row reduced echelon form or neither. Also determine which variables are free if it's in row echelon form or row reduced echelon form.
$\begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ is neither row-echelon from nor reduced row-echelon form (because the leading 1 in the third row is not to the right of the leading 1 in the second row) .
$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ is in row-echelon from. Free variables is x_3.
$\begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ is neither row-echelon from nor reduced row-echelon form (because the leading coefficient in first row is not 1.)
$\begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ is in row reduced echelon from. Free variables is x_2.

2. Determine if the following systems are consistent and if so give all solutions in parametric vector form.
(a)

$$\begin{aligned} x_1 \quad -2x_2 &= 3 \\ 2x_1 \quad -7x_2 &= 0 \\ -5x_1 \quad +8x_2 &= 5 \end{aligned}$$

Solution: The augmented matrix is
$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -7 & 0 \\ -5 & 8 & 5 \end{bmatrix} \sim \text{(new row 2:=-2 row 1+ row 2)} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ -5 & 8 & 5 \end{bmatrix}$$

$\sim$ (new row 3:=5row 1+row 3) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & 20 \end{bmatrix}$

$\sim$ (new row 2:=$\frac{1}{3}$row 2) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 20 \end{bmatrix}$

$\sim$ (new row 3:=2 row 2 + row 3) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 24 \end{bmatrix}$. The last row implies that $0 = 24$ which is impossible. So this system is inconsistent.

(b)

$$\begin{array}{lllll} x_1 & +2x_2 & -3x_3 & +x_4 & = 1 \\ -x_1 & -2x_2 & +4x_3 & -x_4 & = 6 \\ -2x_1 & -4x_2 & +7x_3 & -x_4 & = 1 \end{array}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix}$ $\sim$ (new row 2:=row 1+row 2) $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ -2 & -4 & 7 & -1 & 1 \end{bmatrix}$ $\sim$ (new row 3:=row 1+2row 3) $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$

$\sim$ (new row 3:=(-1)row 2+row 3) $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$ $\sim$ (new row 1:=(-1)row 3+row 1) $\begin{bmatrix} 1 & 2 & -3 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$ $\sim$ (new row 1:=3row

2+row 1) $\begin{bmatrix} 1 & 2 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$. So x_2 is free. The solution is $x_1 = 26 - 2x_2$, $x_3 = 7$, $x_4 = -47$. Its parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26-2t \\ t \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ 7 \\ -4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c)

$$\begin{array}{lllll} x_1 & +2x_2 & -3x_3 & +x_4 & = 1 \\ -x_1 & -2x_2 & +4x_3 & -x_4 & = 6 \\ -2x_1 & -4x_2 & +7x_3 & -2x_4 & = 1 \end{array}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ -2 & -4 & 7 & -2 & 1 \end{bmatrix}$

-2 * Row 2 + Row 3 := new Row 3

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -2 & 4 & -1 & 6 \\ 0 & 0 & -1 & 0 & -11 \end{bmatrix}$$

1 * Row 1 + Row 2 := new Row 2

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & -1 & 0 & -11 \end{bmatrix}$$

1 * Row 2 + Row 3 := new Row 3

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

The last row means that $0 = -4$ which is impossible. So this system is inconsistent.

3. Let $A = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 2 & 6 & 5 & 1 \\ 3 & 9 & 4 & 5 \end{bmatrix}$.

(a) Find all the solutions of the non-homogeneous system $Ax = b$, and write them in parametric form, where $b = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$. Solution:

Consider the augmented matrix $[A\ b] = \begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix}$.

Now we perform row operations on the augmented matrix.

$\begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 2 & 6 & 5 & 1 & -2 \\ 3 & 9 & 4 & 5 & -3 \end{bmatrix}$ $newrow2 := (-2)row1 + row2, newrow3 := (-3)row1 + row3$

$\begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 13 & -13 & 0 \\ 0 & 0 & 16 & -16 & 0 \end{bmatrix}$

$newrow2 := \frac{1}{13}row2, newrow3 := \frac{1}{16}row3, newrow3 := row3 - row2$

$\begin{bmatrix} 1 & 3 & -4 & 7 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$newrow1 := 4row2 + row1$ $\begin{bmatrix} 1 & 3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So the solution is

$$\begin{cases} x_1 + 3x_2 + 3x_4 = -1 \\ x_3 - x_4 = 0 \\ x_2 \text{ and } x_4 \text{ are free.} \end{cases} \tag{1}$$

So

$$\begin{cases} x_1 = -1 - 3x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = x_4 \\ x_4 \text{ is free.} \end{cases} \tag{2}$$

Thus the solution of $Ax = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1-3s-3t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$ where s and t are any numbers.

(b) Find all the solutions of the homogeneous system $Ax = 0$, and write them in parametric form.

Solution: Consider the augmented matrix $[A\,0] = \begin{bmatrix} 1 & 3 & -4 & 7 & 0 \\ 2 & 6 & 5 & 1 & 0 \\ 3 & 9 & 4 & 5 & 0 \end{bmatrix}$.

From previous example, it can be row reduced to $\begin{bmatrix} 1 & 3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So the solution is

$$\begin{cases} x_1 + 3x_2 + 3x_4 = 0 \\ x_3 - x_4 = 0 \\ x_2 \text{ and } x_4 \text{ are free.} \end{cases} \tag{3}$$

So

$$\begin{cases} x_1 = 0 - 3x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = x_4 \\ x_4 \text{ is free.} \end{cases} \tag{4}$$

So the solution of $Ax = 0$ is of the form

$$x = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where s and t are any numbers.

4. Let $S = Span\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -4 \end{bmatrix} \right\}.$

Solution: Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & -3 & 1 \\ 3 & 1 & 2 & 1 \\ 1 & -2 & 3 & -4 \end{bmatrix}$. In part (a), we need to determine if $Ax = v$ is consistent or not. In part (b), we need to determine if $Ax = w$ is consistent or not.

We can just consider the augmented matrix $[A\ v\ w]$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ -2 & 1 & -3 & 1 & 3 & 3 \\ 3 & 1 & 2 & 1 & -2 & -2 \\ 1 & -2 & 3 & -4 & 1 & 1 \end{bmatrix}$$

2 * Row 1 + Row 2 $\rightarrow$ new Row 2

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & \\ 3 & 1 & 2 & 1 & -2 & -2 \\ 1 & -2 & 3 & -4 & 1 & 1 \end{bmatrix}$$

-3 * Row 1 + Row 3 $\rightarrow$ new Row 3

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 5 \\ 0 & 1 & -1 & 1 & 1 & -5 \\ 1 & -2 & 3 & -4 & 1 & 1 \end{bmatrix}$$

-1 * Row 1 + Row 4 $\rightarrow$ new Row 4

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ & & & & & \\ 0 & 1 & -1 & 1 & 1 & -5 \\ 0 & -2 & 2 & -4 & 2 & 0 \end{bmatrix}$$

-1 * Row 2 + Row 3 $\rightarrow$ new Row 3

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -10 \\ 0 & -2 & 2 & -4 & 2 & 0 \end{bmatrix}$$

2 * Row 2 + Row 4 $\rightarrow$ new Row 4

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & -2 & 4 & 10 \end{bmatrix}$$

swap Row 3 with Row 4

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 5 \\ 0 & 0 & 0 & -2 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & -10 \end{bmatrix}$$

-1/2 * Row 3 $\rightarrow$ new Row 3

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 & -10 \end{bmatrix}$$

-1 * Row 3 + Row 2 $\rightarrow$ new Row 2

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 3 & 10 \\ 0 & 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 & -10 \end{bmatrix}$$

This implies the system $Ax = v$ is consistent and v is a linear combination of the vectors in S Similarly, the system $Ax = w$ is inconsistent and w is a not linear combination of the vectors in S

5. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ -1 & 2 & a & 1 \end{array}\right]$$

   (a) For what values of a will the system have a unique solution? What is the solution?(your answer may involve a

   Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ -1 & 2 & a & 1 \end{array}\right]$$

(-1)row 1 + row 2 $\to$ row 2

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2+b \\ -1 & 2 & a & 1 \end{array}\right]$$

row 1 + row 3 $\to$ row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2+b \\ 0 & 3 & a+1 & 3 \end{array}\right]$$

-3 * row 2 + row 3 $\to$ row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2+b \\ 0 & 0 & a+1 & 9+3b \end{array}\right]$$

From this, we know that if $a+1 \neq 0$ then this system has a unique solution. We can continue the row reduction.
$\frac{1}{a+1}$ row 3 $\to$ new row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2+b \\ 0 & 0 & 1 & \frac{9+3b}{a+1} \end{array}\right]$$

(-1) row 3 + row 1 $\to$ new row 1

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2-\frac{9+3b}{a+1} \\ 0 & 1 & 0 & -2+b \\ 0 & 0 & 1 & \frac{9+3b}{a+1} \end{array}\right]$$

(-1) row 2 + row 1 $\to$ new row 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4-b-\frac{9+3b}{a+1} \\ 0 & 1 & 0 & -2+b \\ 0 & 0 & 1 & \frac{9+3b}{a+1} \end{array}\right]$$

Hence if $a \neq -1$ then this system has a unique solution $x_1 = 4-b-\frac{9+3b}{a+1}$, $x_2 = -2+b$ and $x_3 = \frac{9+3b}{a+1}$.

(b) For what values of a and b will the system have infinitely many solutions?

Solution: From previous problem, we know that

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ -1 & 2 & a & 1 \end{array}\right]$$

is row reduced to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2+b \\ 0 & 0 & a+1 & 9+3b \end{array}\right],$$

we know that if $a+1=0$ and $9+3b=0$ then this system has infinitely many solutions.

(c) For what values of a and b will the system be inconsistent? Solution: From previous problem, we know that

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & b \\ -1 & 2 & a & 1 \end{array}\right]$$

is row reduced to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2+b \\ 0 & 0 & a+1 & 9+3b \end{array}\right],$$

we know that if $a+1=0$ and $9+3b \neq 0$ then this system is inconsistent.

6. (a) Find the inverses of the following matrices if they exist.

$A = \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$.

Solution: Consider the augmented matrix $[A\ I_2]$. $\begin{bmatrix} 7 & -2 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{bmatrix}$

2 * Row 2 + Row 1 $\rightarrow$ new Row 1

$\begin{bmatrix} -1 & 0 & 1 & 2 \\ -4 & 1 & 0 & 1 \end{bmatrix}$

-1 * Row 1 $\to$ new Row 1

$\begin{bmatrix} 1 & 0 & -1 & -2 \\ -4 & 1 & 0 & 1 \end{bmatrix}$

4 * Row 1 + Row 2 $\to$ new Row 2

$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -4 & -7 \end{bmatrix}$ So $A^{-1} = \begin{bmatrix} -1 & -2 \\ -4 & -7 \end{bmatrix}$.

Consider the augmented matrix $[B\ I_3]$. $\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

-1 * Row 1 + Row 2 $\to$ new Row 2

$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

-1 * Row 1 + Row 3 $\to$ new Row 3

$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix}$

1/2 * Row 2 $\to$ new Row 2

$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix}$

-1 * Row 2 + Row 3 $\to$ new Row 3

$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & -1 & -1/2 & -1/2 & 1 \end{bmatrix}$

-1 * Row 3 $\to$ new Row 3

$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{bmatrix}$

-1 * Row 3 + Row 1 $\to$ new Row 1

$$\begin{bmatrix} 1 & -1 & 0 & 1/2 & -1/2 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{bmatrix}$$

1 * Row 2 + Row 1 $\rightarrow$ new Row 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{bmatrix} \text{ So } B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}$$

Now consider $[C\ I_3]$. $\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 5 & 6 & 7 & 0 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{bmatrix}$

-2 * Row 1 + Row 2 $\rightarrow$ new Row 2

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 & 1 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{bmatrix}$$

Row 1 swap Row 2

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{bmatrix}$$

-2 * Row 1 + Row 2 $\rightarrow$ new Row 2

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 5 & -2 & 0 \\ 8 & 9 & 10 & 0 & 0 & 1 \end{bmatrix}$$

-8 * Row 1 + Row 3 $\rightarrow$ new Row 3

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 5 & -2 & 0 \\ 0 & 9 & 18 & 16 & -8 & 1 \end{bmatrix}$$

1/3 * Row 2 $\rightarrow$ new Row 2

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 5/3 & -2/3 & 0 \\ 0 & 9 & 18 & 16 & -8 & 1 \end{bmatrix}$$

-9 * Row 2 + Row 3 $\rightarrow$ new Row 3

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 5/3 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Thus C is not invertible.

(b) What's $(B^T)^{-1}$?

Solution: $(B^T)^{-1} = (B^{-1})^T = (\begin{bmatrix} 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix})^T = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & -1 \end{bmatrix}$.

7. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $f(x) = x^2 - 2x + 2$. Show that $f(A) = 0_2$. Here 0_2 is the 2×2 zero matrix.

   Solution: First, we compute $A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. So $f(A) = A^2 - 2A + 2I_2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   $= \begin{bmatrix} 0-2+2 & -2+2+0 \\ 2-2+0 & 0-2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

8. Let A be an 3×3 matrix. Suppose $A^3 + 2A^2 - 4A + I_3 = 0$. Is A invertible? Express A^{-1} in terms of A if possible.

   Solution: From $A^3 + 2A^2 - 4A + I_3 = 0$, we have $-A^3 - 2A^2 + 4A = I_3$ and $A(-A^2 - 2A + 4I_3) = I_3$. This implies that A is invertible and $A^{-1} = -A^2 - 2A + 4I_3$.

9. Express the following matrices as a product of elementary matrices and a matrix in reduced row-echelon form.

   $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

   Solution:

   (a) Step 1. $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

   -1 * Row 1 + Row 2 $\rightarrow$ new Row 2

   $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

   Repeat the same operation on the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

-1 * Row 1 + Row 2 $\rightarrow$ new Row 2

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2. 1 * Row 1 + Row 3 $\rightarrow$ new Row 3

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Repeat the same operation on the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1 * Row 1 + Row 3 $\rightarrow$ new Row 3

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 3.

Row 2 $\leftrightarrow$ Row 3

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Repeat the same operation on the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Row 2 $\leftrightarrow$ Row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 4.

1/2 * Row 2 $\rightarrow$ new Row 2

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Repeat the same operation on the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1/2 * Row 2 $\rightarrow$ new Row 2

$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 5.

-1 * Row 2 + Row 1 $\rightarrow$ new Row 1

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Repeat the same operation on the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

-1 * Row 2 + Row 1 $\rightarrow$ new Row 1

$E_5 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Finally

$A = E_5 E_4 E_3 E_2 E_1 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

(b)

step 1. $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

-4 * Row 1 + Row 2 $\rightarrow$ new Row 2

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}$

Repeat the same operation on the identity matrix I_2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

-4 * Row 1 + Row 2 $\rightarrow$ new Row 2

$E_1 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

step 2.

-1/3 * Row 2 $\rightarrow$ new Row 2

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Repeat the same operation on the identity matrix I_2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

-1/3 * Row 2 $\rightarrow$ new Row 2

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$

step 3.

-2 * Row 2 + Row 1 $\rightarrow$ new Row 1

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Repeat the same operation on the identity matrix I_2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

-2 * Row 2 + Row 1 $\rightarrow$ new Row 1

$E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

So $B = E_3E_2E_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$.