Meromorphic continuation of the scattering matrix in the Stark effect case*

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November 21, 2003

Abstract

The scattering matrix that describes the scattering of a quantum mechanical particle in $\mathbf{R}^n$ by a short range potential in the presence of a constant electric field is considered. The scattering matrix is shown to have a meromorphic continuation in the energy variable to the whole complex plane as a bounded operator on $L^2(\mathbf{R}^{n-1})$. The constant electric field, which is referred to as the "Stark effect" in this context, means that "short range" includes the physically interesting case of the Coulomb potential (but $n \geq 3$ is required here.) In addition it is shown that the resolvent operator has a meromorphic continuation across the real axis in an appropriate context. The scattering matrix studied here is constructed using Isozaki, Kitada and Yajima's "time independent modifiers."

1 Introduction.

It is shown here that the scattering matrix has a meromorphic extension in the energy variable in the following physical context. A single (nonrelativistic) quantum particle is scattered by a potential in the presence of a constant electric field ("Stark effect"). The potentials V must be short range but, because of the constant field that includes, for example the Coulomb potential, $V(x) = C/|x|$ where C is a real constant. (However the potential must not

*AMS Subject Classification Primary 35P25 Secondary 35J10, 47A40, 47F05, 81U20

be too locally singular and so the results below apply to the Coulomb potential only when the space dimension n is at least 3). The results here were over-viewed in [13].

The Schrödinger operators operators that model the above physical experiment are introduced as follows. Let V be the potential and $F > 0$ be the (constant) field strength. Assuming the field acts in the $-\mathbf{e}_1$ direction where $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n$, the Schrödinger operators are (after convenient normalizations)

$$H_0 = -\Delta + Fx_1 \quad \text{and} \quad H = H_0 + V$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$. The precise assumptions on the potential V are given in the Hypotheses below but $V(x)$ is short range in the sense that the usual "Møller wave operators" exist and are complete. However in this paper we work instead with the two Hilbert space wave operators, or "time independent modifiers" those of Isozaki, Kitada and Yajima [17, 18, 22, 23]

$$W_J^{\pm} = \operatorname{s-}\lim_{t \to \pm\infty} e^{itH} J e^{-itH_0} \tag{1.1}$$

where J is a bounded operator on $L^2(\mathbf{R}^n)$ to be chosen appropriately and "s-lim" means the limit is taken in the strong topology. ($W_J^{\pm}$ exist and are complete [32], assuming the Hypotheses below.) Of course the Møller wave operators, referred to above, are just $W_J^{\pm}$ when J is the identity operator. This reliance on $W_J^{\pm}$ is also the starting point for Gérard and Martinez [9] who established the existence of a meromorphic continuation of the scattering matrix in the absence of the Stark effect ($F = 0$). Indeed [9] was the inspiration for the present work; however Gérard and Martinez are able to treat long range potentials unlike here. Isozaki and Kitada [19] were the first to use $W_J^{\pm}$ to study the scattering matrix particularly smoothness of the scattering amplitude although they didn't address the question of meromorphic continuation. Regrettably the relationship between $W_J^{\pm}$ and the Møller wave operators (when both exist) is not obvious but see [21] and [8, §4.9] when $F = 0$ and [33] for $F \neq 0$. This work concerns only $W_J^{\pm}$ and the corresponding scattering operator $S = (W_J^+)^* W_J^-$; the Møller wave operators will be the object of a future study.

Agmon and Klein [2] have also established the existence of a meromorphic extension for the scattering matrix and without reference to the two Hilbert space wave operators. Indeed they consider very long range potentials but the potentials are spherically symmetric (and $F = 0$). It is intriguing that in the two works treating long range potentials, [2] and [9], the continuation of the scattering matrix is, as operators on certain spaces which are Gevrey spaces where the two works overlap. Which Gevrey spaces are appropriate depends on the rate of decay of the potential. Since the potentials here are short range, there is no question about where to extend the matrix: the continuation is,as a bounded operator on $L^2(\mathbf{R}^{n-1})$. The author's attempt to extend the results here to general long range potentials (like those in [32]) was problematic.

In the "Stark effect" case ($F \neq 0$) Yajima [34] has established the existence of a meromorphic continuation of the scattering matrix (again without reference to the two Hilbert space wave operators). He considers potentials which decrease exponentially fast in the field direction ($-\mathbf{e}_1$) (corresponding roughly to $V_{\mathcal{A}} = 0$ below). Several of the arguments below are adaptations of Yajima's to the present context.

Kvitsinsky and Kostrykin [26] show that the scattering matrix has a meromorphic extension in the case V is the Coulomb potential and $n = 3$. They use Jost functions and parabolic coordinates.

In the case of N-body scattering and $F = 0$ the existence of a meromorphic extension of the scattering matrix has been treated by several authors: see Hagedorn [10, 1979], Sigal [31, 1986], for example in the short range case and Bommier [5, 1994] in the long range case.

Introduce now the assumptions on the potential V. The following notation will be convenient when discussing exponential decay in the direction of the field: for $\mu > 0$ define

$$h_\mu(x_1) = e^{\mu x_1}\tilde{\chi}_{(-\infty,0)}(x_1) + \tilde{\chi}_{(0,\infty)}(x_1) \tag{1.2}$$

and $\tilde{\chi}_{(-\infty,0)}$ is in $C^\infty(\mathbf{R})$

$$\begin{aligned} \tilde{\chi}_{(-\infty,0)}(x_1) &= \begin{cases} 1 & \text{if } x_1 < -1/2 \\ 0 & \text{if } x_1 > 1/2 \end{cases} \qquad \text{and} \\ \tilde{\chi}_{(0,\infty)}(x_1) &= \tilde{\chi}_{(-\infty,0)}(-x_1) = 1 - \tilde{\chi}_{(-\infty,0)}(x_1). \end{aligned} \tag{1.3}$$

In addition $C^\infty(\Omega)$ denotes the set of all infinitely differentiable functions defined on $\Omega \subseteq \mathbf{R}^n$; and $\langle x_1 \rangle = (1 + x_1^2)^{1/2}$.

Hypotheses: $V = V_{\mathcal{A}} + V_e$ *where* $V_{\mathcal{A}}(x)$ *is* $C^\infty(\mathbf{R}^n)$, *real valued and has an analytic extension to the cone*

$$\{x \in \mathbf{C}^n : \Re x_1 < -R_V, |\Im x| < \rho_V |\Re x_1|\} \tag{1.4}$$

for some $R_V > 0$ and $\rho_V > 0$, and for some positive constants C and ϵ_V

$$\begin{aligned} |V_{\mathcal{A}}(x)| &\le C\langle \Re x_1\rangle^{-\epsilon_V} && (1.5)\\ \lim_{|x|\to\infty} V_{\mathcal{A}}(x) &= 0. && (1.6)\end{aligned}$$

The other term is $V_e = h_{\mu_V}(x_1)V_0$ where V_0 is symmetric and H_0-compact and commutes with any operator which is multiplication by a function of x_1 and $\mu_V > 0$.

The Hypotheses will be assumed throughout; for the main results the additional assumption $\epsilon_V > 1/2$, which makes V short range with respect to H_0, will be required. However for the construction of the operator J, in §2, 3 and 4, the assumption $\epsilon_V > 0$ is adequate so that V may be long range. The Hypotheses imply that V is H_0-compact and this assures that H is self adjoint. For an introduction to Schödinger operators with Stark effect, including examples of H_0-compactness, see [28].

Example: The potentials $V(x) = C\langle x\rangle^{-\epsilon_V}$ satisfy the Hypotheses if C is a real constant and $0 < \epsilon_V$. Also the Coulomb potential $V(x) = C/|x|$ satisfies the Hypothesis provided $n \ge 3$ (to assure H_0-compactness).

Remark: The assumptions on $V_{\mathcal{A}}$ assure that the derivatives satisfy the same type of bounds or more precisely

$$\begin{aligned} |D_x^\alpha V_{\mathcal{A}}(x)| &< C_\alpha\langle \Re x_1\rangle^{-|\alpha|-\epsilon_V}\\ \lim_{|x|\to\infty} D_x^\alpha V_{\mathcal{A}}(x) &= 0\end{aligned}$$

($\alpha \in \mathbf{N}_0^n$) on a set a slightly smaller than (1.4),

$$\{x \in \mathbf{C}^n : \Re x_1 < -R, |\Im x| < \rho|\Re x_1|\}$$

($R > R_V$; $\rho < \rho_V$). This follows from Cauchy's integral formula, appropriately differentiated. Observe further that the breakdown of $V = V_{\mathcal{A}}+V_e$ is not unique. It will be convenient to cut off $V_{\mathcal{A}}(x)$ so that it is 0 if $\Re x_1 > -R+1$ but it remains unchanged if $\Re x_1 < -R$. for then

$$|D_x^\alpha V_{\mathcal{A}}(x)| < C_\alpha R^{-(\epsilon_V-\epsilon_1)}\langle \Re x_1\rangle^{-|\alpha|-\epsilon_1} \qquad (1.7)$$

for any ϵ_1 $0 < \epsilon_1 < \epsilon_V$. Later R will be chosen large enough to assure that finitely many of these derivatives can be made appropriately small. To simplify notation, these estimates will be assumed to hold on the original set (1.4): $R = R_V$, $\rho = \rho_V$. These estimates are stronger than those assumed in [32] and it follows from there that the two Hilbert space wave operators exist and are complete. (The choice of J here is different but the proof is very similar.)

To state the main result of this paper, some notation is needed. The scattering operator is defined by $S = (W_J^+)^* W_J^-$. The scattering matrix $S(\lambda)$ is defined by restricting S to manifolds (hyper-planes in this context) of constant free energy λ. The restriction operator, $T_0(\lambda)$ is constructed explicitly in §5; it is a mapping from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{n-1})$ and formally $S(\lambda)T_0(\lambda) = T_0(\lambda)S$. Let

$$V_J = HJ - JH_0$$

be the "effective potential" and $R(z) = (H - z)^{-1}$ be the resolvent. The main result of this paper is

Theorem 1 *Assume $\epsilon_V > 1/2$. For almost every real λ*

$$S(\lambda) - 1 = -2\pi i T_0(\lambda) J^* V_J T_0(\lambda)^* + 2\pi i T_0(\lambda) V_J^* R(\lambda + i0) V_J T_0(\lambda)^*. \quad (1.8)$$

Additionally, $S(\lambda)$ has a meromorphic extension in λ to all of $\mathbf{C}$ as a bounded operator on $L^2(\mathbf{R}^{n-1})$.

Here $R(\lambda + i0) = \lim_{\mu \to 0+} R(\lambda + i\mu)$ which exists in a context to be made precise in §6. In fact most of the work proving Theorem 1 is not verifying (1.8) but showing that the resolvent has a meromorphic continuation, in a sense to be clarified by Theorem 6.2. Additionally it is shown that the first term on the right side of (1.8) has an entire extension which implies that any pole of (the extension of) $S(\lambda)$ is a pole of the (extended) resolvent. Poles of the resolvent are known as *resonances*. The converse question of whether every resonance is a pole of $S(\lambda)$ is not addressed here. Gérard and Martinez [9] prove the comparable result in the case $F = 0$. They show if one adopts Hunziker's [16] method of extending the resolvent (a version of "complex scaling") then a pole of the resolvent is a pole of $S(\lambda)$. This approach encounters difficulties in the case $F \neq 0$ because the spectral deformation technique of [16] causes the spectrum of H_0 to vanish completely; see Herbst [11, 1979]. A different approach to extending the resolvent is taken here: see Theorem 6.2.

The plan of this paper is to begin with the construction of the operator J as a Fourier integral operator. This construction is fundamental to the entire paper because it assures that the symbol of the effective potential V_J decays exponentially fast in the field direction at least in the incoming and outgoing regions of phase space. Exponential decay of the potential is what Yajima [34] required for his continuation result. The phase is constructed in §2; the symbol in §3. The construction of the symbol imitates Gérard and Martinez's [9] construction and requires considerable care; the phase's construction is relatively crude. As in [9] the symbol fails to be analytic in x_1 but it is "almost analytic" in a sense to be clarified in §4. In §5, $T_0(\lambda)$, referred to above, is formally defined and conditions on Fourier integral operators Q are derived that assure $QT_0(\lambda)^*$ has an entire extension. In §6 the resolvent of H_0 is shown to have an analytic extension across the real axis in the appropriate setting and similarly it is shown that $R(z)$ has a meromorphic extension across the real axis. The arguments in §6 are analogues of Yajima's [34] and do not use "complex scaling" as was done in [9]. Therefore this work parallels Gérard and Martinez's [9] in the construction of J especially in §3 and §4; the later sections are more closely related to Yajima's [34]. In §7 the proof of Theorem 1 is given.

Notation: Below is a brief list of the less standard notation used. Equation numbers refer to the equation *closest* to where the notation is introduced.

$a, a_{\mathcal{A}}, a_e$ see after (4.5)	b see Proposition 3.1 and (4.2)
$b_{\mathcal{A}}, b_e$ see before (4.4)	$\mathcal{B}, \mathcal{B}_{\mathcal{A}}, \mathcal{B}(M,\delta), \mathcal{B}_{\mathcal{A}}(M,\delta)$ see before (5.17)
$D_j = -i\partial/\partial x_j$	$d_1x = (2\pi)^{-n/2}dx$, defined prior to (2.1)
$\partial_{x_j}, \partial_{\overline{x}_j}$ see 3.2	$G(\xi) = (1/3)\xi_1^3 + \xi_1\lvert\xi_\perp\rvert^2$ see (5.1)
h_μ see (1.2)	$H_{\mathcal{A}} = H_0 + V_{\mathcal{A}}$ see before (2.1)
m_1, before Lemma 3.4	m_2 after Lemma 3.4
$T_0(\lambda)$ see prior to (5.3)	U see (5.2)
$\Gamma(\gamma, R, K, \kappa)$ see (2.4)	$\Gamma_{\Omega,\delta}$ see (4.1)
ϕ see Proposition 2.1	$\xi_\perp$ see after (2.2)
$\Omega_{R,\delta}$ see after (5.16)	$\langle x\rangle = (1 + x\cdot x)^{1/2}$ see before (3.11)
$\lVert\cdot\rVert$ is the operator norm	$\lvert\cdot\rvert_k$ see (5.6) $\lVert\cdot\rVert_m$ see (5.17)
$(\cdot,\cdot)$ is the L^2 inner product	

Also χ_A denotes the characteristic function of a set A and $\tilde{\chi}_A$ is a C^∞ approximation of χ_A.

2 The Phase.

In this and the next section the operator J of (1.1) will be constructed. The Fourier transform $\mathcal{F}$ will be normalized as

$$\hat{u}(\xi) \equiv \mathcal{F}u(\xi) = \int e^{-ix\cdot\xi}u(x)d_1x$$

where d_1 is $(2\pi)^{-n/2}$ times Lebesgue measure on $\mathbf{R}^n$, and where u is in the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ of C^∞ functions of rapid decrease. Following Isozaki, Kitada and Yajima [18, 19, 22] J is defined as

$$Ju(x) = \int e^{i\phi(x,\xi)}a(x,\xi)\hat{u}(\xi)\, d_1\xi.$$

where the phase function ϕ is to be constructed in this Section and the symbol a in the next Section. (Although J is a Fourier integral operator, little knowledge of Fourier integral operators is assumed here. Needed results will be stated and will concern primarily pseudo-differential operators.) The construction is, to some extent arbitrary since the phase and symbol are not uniquely determined by the Fourier integral operator and there is some flexibility in the choice of J. The construction of the phase is not exactly parallel to the construction of Isozaki and Kitada [18] in the case $F = 0$ although imitating their construction is a viable alternative; instead a closely related method of [32] is used and the reader is referred there for additional details.

The objective in the choice of the phase and symbol both is to assure that the symbol t of the operator $H_{\mathcal{A}}J - JH_0$ where $H_{\mathcal{A}} = H_0 + V_{\mathcal{A}}$ decays exponentially in x_1 at least in the incoming and outgoing regions of phase space. This choice is analogous to that of Gérard and Martinez [9]. Explicitly t is

$$\begin{aligned} t(x,\xi) \;&=\; -2i\nabla_x\phi(x,\xi)\cdot\nabla_x a(x,\xi) + iF\frac{\partial a}{\partial \xi_1}(x,\xi) \\ &\quad + p(x,\xi)a(x,\xi) - \Delta_x a(x,\xi) \end{aligned} \tag{2.1}$$

where

$$p(x,\xi) = |\nabla_x\phi(x,\xi)|^2 - F\frac{\partial\phi}{\partial\xi_1}(x,\xi) + Fx_1 + V_{\mathcal{A}}(x) - |\xi|^2 - i\Delta_x\phi(x,\xi) \tag{2.2}$$

The choice of ϕ should assure that $p(x,\xi) \to 0$ as $\Re x_1 \to -\infty$ as rapidly as possible. Actually instead of the phase function ϕ it is more convenient to construct the restriction ϕ^+ of ϕ to an outgoing space $\{\xi : \Re\xi_1 < \kappa\}$ for some constant κ, to be specified below. The incoming part can then be constructed similarly or, as here, by "time reversal" and ϕ is formed by gluing the two parts together (4.6). In this Section the discussion is entirely for the outgoing "+" case and so frequently the "+" superscripts are omitted. Thus ϕ^+ should be an approximate solution of the differential equation $p = 0$. In addition θ^+, defined by $\phi^+(x,\xi) = x\cdot\xi + \theta^+(x,\xi)$, should satisfy the "boundary condition": $\theta^+(x,\xi) \to 0$ as $\Re x_1 \to -\infty$, again when $\Re\xi_1 < \kappa$. In this construction of ϕ^+ the term $-i\Delta_x\phi$ in (2.1) is neglected; $\phi^+(x,\xi)$ will be real when (x,ξ) are.

The first approximation of θ^+ is the solution θ_1 of

$$2\xi \cdot \nabla_x \theta_1 - F\frac{\partial\theta_1}{\partial\xi_1} = -V_{\mathcal{A}}$$

given by, integrating along the characteristic curves

$$\theta_1(x,\xi) = \int_0^\infty V_{\mathcal{A}}(x + 2t\xi - Ft^2\mathbf{e}_1) - V_{\mathcal{A}}(-R\mathbf{e}_1 + 2t\xi_\perp - Ft^2\mathbf{e}_1)\,dt$$

where $\xi_\perp = (0, \xi_2, \ldots, \xi_n)$ and $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbf{C}^n$ and $R > R_V$ (R_V as in the Hypotheses of §1). The second term in the integrand assures that the integral exists as an improper Riemann integral; in fact the integral converges locally uniformly in (x,ξ) (see (1.7)) so that θ_1 is differentiable and its derivatives can be computed by differentiating "under the integral sign". The second and final approximation of θ^+ is $\theta_2 = \theta_1 + \theta^\Delta$ where θ^Δ is a correction term defined as follows. Introduce a cutoff function $\tilde{\chi}_{(-\infty,1)}$, a $C^\infty(\mathbf{R})$ function (that is infinitely differentiable) so that

$$\tilde{\chi}_{(-\infty,1)}(x_1) = \begin{cases} 1 & \text{if } x_1 < 1/2 \\ 0 & \text{if } x_1 > 1 \end{cases}$$

Further define

$$\psi(x,\xi) \quad = \quad \tilde{\chi}_{(-\infty,1)}(\Re x_1)\tilde{\chi}_{(-\infty,1)}(\Re\xi_1)|\nabla_x\theta_1(x,\xi)|^2.$$

The correction term is the solution of

$$2\xi \cdot \nabla_x \theta^\Delta - F\frac{\partial\theta^\Delta}{\partial\xi_1} = -\psi$$

given by

$$\theta^{\Delta}(x,\xi) = \int_0^\infty \psi(x + 2t\xi - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1)\, dt$$

(The construction in [32] needed additional corrections because of the weaker assumptions there. Further corrections θ_j, $j = 3, 4, 5, \ldots$ are obviated by the construction of the symbol in the next Section.) Define θ^+ by cutting off θ_2 as follows: let $\tilde{\chi}_{(-\infty,-R)}$ be a $C^\infty(\mathbf{R})$ function which is $\tilde{\chi}_{(-\infty,-R)}(x_1) = 0$ if $x_1 > -R$ and $\tilde{\chi}_{(-\infty,-R)}(x_1) = 1$ if $x_1 < -R - 1$. Let

$$\omega_0(x_1) = (x_1 + R + 1)\tilde{\chi}_{(-\infty,-R)}(x_1) \; - \; R - 1$$

and

$$\theta^+(x_1, x_\perp, \xi) = \theta_2(\omega_0(x_1), x_\perp, \xi). \tag{2.3}$$

This cutoff locks in the analyticity of θ^+ in the other variables when $x_1 > -R$ is fixed. This completes the construction of θ^+ and therefore of $\phi^+(x,\xi) = x \cdot \xi + \theta^+(x,\xi)$.

Recall that $V_{\mathcal{A}}$ extends to the cone (1.4) analytically. Correspondingly the phase $\phi^+(x,\xi)$ and symbol will be analytic on certain truncated cones:

$$\begin{aligned}\Gamma(\gamma, R, K, \kappa) \;\; = \;\; & \{(x,\xi) \in \mathbf{C}^n \times \mathbf{C}^n : |\Im x| < -\gamma\rho_V(\Re x_1 + K), \Re x_1 < -R, \\ & |\Im\xi| < \max\{-\gamma^2\rho_V \Re\xi_1, |\kappa|\}, -\Re\xi_1 > \kappa\}\end{aligned} \tag{2.4}$$

where $0 < \gamma \leq 1$, κ is real. The parameters in this definition will be chosen as appropriate during the construction of the symbol and phase but intuitively $1 - \gamma$, $1/R$, $1/K$ will be small positive constants and $-1 < \kappa < 1$ is negative in this Section and positive in the next. The properties of ϕ^+ are summarized in the Proposition below. Introduce the notation

$$C_b^\infty(\Omega)$$

for the set of all $C^\infty(\Omega)$ functions which are bounded along with all their partial derivatives on a subset Ω of Euclidean space.

Proposition 1 *Let $0 < \gamma_\phi$, $\kappa_\phi < 1$, $\delta > 0$ and $k \in \mathbf{N}$ be given constants. Then there exist R_ϕ and K_ϕ, $0 < K_\phi < R_\phi$ so that the phase, $\phi^+(x,\xi)$, as defined above on*

$$\Omega = \Gamma(\gamma_\phi, R_\phi, K_\phi, -\kappa_\phi) \cup \{(x,\xi) \in \mathbf{R}^{2n} : \xi_1 < \kappa_\phi\} \tag{2.5}$$

is real valued when (x,ξ) *are and holomorphic on* $\Gamma \equiv \Gamma(\gamma_\phi, R_\phi, K_\phi, -\kappa_\phi)$. *Additionally, if* $x_1 \geq -R_\phi$ *is fixed then* $\phi^+(x,\xi)$ *is holomorphic in the variables* $(x_\perp, \xi)$ *on the cross section* $\{(x_\perp,\xi) : (-R-1, x_\perp, \xi) \in \Gamma\}$ *of* Γ *for some* $R > K_\phi$. *Further,*

$$\sup\{|D_x^\alpha D_\xi^\beta[\nabla_x \phi^+(x,\xi) - \xi]| : |\alpha| + |\beta| \leq k, (x,\xi) \in \Omega\} < \delta. \tag{2.6}$$

Moreover for any ϵ_ϕ, $0 < \epsilon_\phi < \epsilon_V$,

$$|\nabla_x \phi^+(x,\xi) - \xi| < C|\Re x_1|^{-\epsilon_\phi}(|\Re x_1| + (\Re \xi_1)^2)^{-1/2}; \tag{2.7}$$

$$|p(x,\xi)| < C|\Re x_1|^{-1-\epsilon_\phi}(|\Re x_1| + (\Re \xi_1)^2)^{-1/2} \tag{2.8}$$

for all $(x,\xi) \in \Gamma$ *and for some* $C > 0$. *The functions* p *and* $\nabla_x \phi^+ - \xi$, *and* $\nabla_\xi \phi^+ - x$,*belong to* $C_b^\infty(\Omega)$ *and for all multi-indices* α, β

$$\lim_{|(x,\xi)| \to \infty} \chi_{(-\infty,R)}(x_1)[|D_x^\alpha D_\xi^\beta[\nabla_x \phi^+(x,\xi) - \xi]| + |D_x^\alpha D_\xi^\beta p(x,\xi)|] = 0$$

($\chi_{(-\infty,R)}$ *denotes the characteristic function of the interval* $(-\infty, R)$.*) Finally, if* $\epsilon_V > 1/2$ *in the Hypothesis, then* $\phi^+(x,\xi) - x \cdot \xi$ *itself is in* $C_b^\infty(\Omega)$ *and* $\phi^+(x,\xi) - x\cdot\xi = O(|\Re x_1|^{-\epsilon_V + 1/2})$ *on* Γ.

Estimates (2.7), (2.8) are typical of those required in this and the next Section and so it is worthwhile to formulate:

Lemma 2 *For any* $\epsilon > 0$ *there is a constant* $C_\epsilon > 0$ *so that for all* $a, c > 0$, $b \geq 0$, $j, k \geq 0$,
(a) if $j + k > 1/2 + \epsilon$ *then*

$$\int_0^\infty (a + 2bt + t^2)^{-k}(c + 2bt + t^2)^{-j}\, dt < \frac{C_\epsilon \max\{\sqrt{a}, \sqrt{c}\}}{\sqrt{j + k - 1/2}} a^{-k} c^{-j}$$

(b) if $k > 1 + \epsilon$ *and* $c = (a + b^2)/2$ *then*

$$\int_0^\infty (a + 2bt + t^2)^{-k}(c + 2bt + t^2)^{-j}\, dt < \frac{C_\epsilon}{\sqrt{k}} a^{-k+1} c^{-j-1/2}.$$

A proof appears in the Appendix.

Outline of the Proof of Proposition 1. Bounds (2.7), (2.8) follow from definitions of θ_1 and θ^Δ and the comparable bounds for $V_\mathcal{A}$ by way of

Lemma 2 (with $a = \Re x_1$ and $b = \Re\xi_1/\sqrt{F}$) in the case $\Re\xi_1 \geq 0$. The case $-\kappa < \Re\xi_1 < 0$ can be reduced to the case $\Re\xi_1 = 0$ by an elementary shift of variables. The arbitrary bound $\delta > 0$ in (2.6) is a consequence of the cutoff of $V_{\mathcal{A}}$ (see 1.7) as well as the cutoff (2.3) of $\theta^+(x,\xi) = \phi^+(x,\xi) - x\cdot\xi$. The decay as $|x_\perp| \to \infty$ is verified in [32, p. 542] and it follows from Lemma 3.3 below.

It remains to check where θ^+ is holomorphic. Observe that any characteristic curve $(x+2t\xi - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1)$ that begins at (x,ξ) in $\Gamma(\gamma, R_1, K_1, -\kappa_\phi)$ (in the domain of $V_{\mathcal{A}}$) remains in $\Gamma(1, R, K, -\kappa_\phi)$ for $t \geq 0$ provided $R_1 - R$ and $K_1 - K$ are large enough and $\gamma < 1$. Since the integrals defining θ_1 and θ^Δ converge locally uniformly, it follows that θ^+ is holomorphic on $\Gamma(\gamma_\phi, R_\phi, K_\phi, -\kappa_\phi)$. As for the holomorphy when $\Re x_1 \geq -R$ is fixed, that is a consequence of the cutoff (2.3) in the definition of θ^+. That completes the proof outline. (More detail may be found in [32, Section 4].), □

Remark: It follows from the Proposition 1 that $e^{i\theta}$ is in C_b^∞ when $\epsilon_V > 1/2$ and therefore could be regarded as part of the symbol of J. This is in marked contrast to the long range case of [32]. However ϕ will be treated as the phase with one exception: Lemma 5.1 below.

3 The Symbol.

The symbol $a(x,\xi)$ of the operator J is constructed in this and the next Section. In this Section it is actually the restriction b of a to an outgoing region of space that is constructed and a is defined in terms of b by (4.4) below. On the one hand b will asymptotically approach 1 as $\Re x_1 \to \infty$. (Later it will be seen that J has a right inverse.) On the other hand b will be chosen so that the symbol t of $H_{\mathcal{A}}J^+ - J^+H_0$ (of equation (2.1)) will decay to 0 exponentially fast as $\Re x_1 \to -\infty$ also in the outgoing regions of phase space. Define

$$t^+(x,\xi) = e^{-i\phi^+(x,\xi)}\left[-\Delta + Fx_1 + V_{\mathcal{A}}(x) - |\xi|^2 - iF\frac{\partial}{\partial\xi_1}\right]e^{i\phi^+(x,\xi)}b(x,\xi) \tag{3.1}$$

or alternately t^+ is given by (2.1) and (2.2) if t, a and ϕ there are replaced by t^+, b and ϕ^+ respectively. Thus t and t^+ agree in a certain outgoing region of phase space. Regrettably b will not be analytic in the x_1 variable, although it is "almost" in the sense that the Cauchy Riemann equations are asymptotically satisfied; see (3.7) below. It will therefore be necessary to be more precise about the derivatives of b in the (complex) x_1 variable. Introduce therefore the notation

$$\partial_{x_j} = \frac{1}{2}\left(\frac{\partial}{\partial \Re x_j} - i\frac{\partial}{\partial \Im x_j}\right) \quad \partial_{\overline{x}_j} = \frac{1}{2}\left(\frac{\partial}{\partial \Re x_j} + i\frac{\partial}{\partial \Im x_j}\right) \tag{3.2}$$

for $1 \leq j \leq n$. The main result of this Section is this.

Proposition 1 *For any γ_σ, $0 < \gamma_\sigma < \gamma_\phi$, $\kappa_\sigma > 0$ there exist positive constants R_σ, K_σ, $R_\sigma > K_\sigma > 0$, and a function $b(x,\xi)$ defined and C^∞ on $\Gamma \equiv \Gamma(\gamma_\sigma, R_\sigma, K_\sigma, \kappa_\sigma)$ such that, for fixed x_1, $\Re x_1 < -R_\sigma$, $b(x,\xi)$ is holomorphic in the remaining $2n-1$ variables on (the cross section of) Γ. Let $k_\sigma \in \mathbf{N}_0$ be arbitrary. Then, for all $j,k \in \mathbf{N}_0$, there is $\mu > 0$, and $C_{j,k} > 0$ so that,*

$$|\partial^j_{\overline{x}_1}\partial^k_{x_1} t^+(x,\xi)| \leq C_{j,k}|\Re\xi_1|^{-k_\sigma} e^{-\mu\langle \Re x_1\rangle}; \tag{3.3}$$

$$\lim_{|(x,\xi)|\to\infty} \partial^j_{\overline{x}_1}\partial^k_{x_1} t^+(x,\xi) = 0; \tag{3.4}$$

$$|\Re x_1|^{j+k+1/2+\epsilon_\phi}(|\Re x_1| + (\Re\xi_1)^2)^{1/2}|\partial^j_{\overline{x}_1}\partial^k_{x_1}(b(x,\xi)-1)| \leq C_{j,k}; \tag{3.5}$$

$$\lim_{|(x,\xi)|\to\infty} |\Re x_1|^{j+k+1/2}(|\Re x_1| + (\Re\xi_1)^2)^{1/2}|\partial^j_{\overline{x}_1}\partial^k_{x_1}(b(x,\xi)-1)| = 0; \tag{3.6}$$

$$|\partial^{j+1}_{\overline{x}_1}\partial^k_{x_1} b(x,\xi)| < C_{j,k} e^{-\mu\langle \Re x_1\rangle}. \tag{3.7}$$

Here γ_ϕ and ϵ_ϕ are constants of Proposition 2.1; $0 < \gamma_\phi < 1$, $0 < \epsilon_\phi < \epsilon_V$ and t^+ was defined by (3.1). The parameter k_σ plays no role below; $k_\sigma = 0$ is adequate for the applications here.

The remainder of this Section is devoted to the proof of the Proposition; the proof parallels that of Gérard and Martinez in [9, §III]. The construction of b is as a formal series, $b \sim \sum_{k\geq 0} b_k$. Each $b_k(x,\xi)$ will be holomorphic on some region like $\Gamma(\gamma, R, K, \kappa)$ so that the x derivatives will behave (at least) like

$$D^\alpha_x b_k = O(\langle \Re x_1\rangle^{-k-|\alpha|}).$$

If one substitutes the formal series for b into the equation $t^+ = 0$ and collects the terms of like asymptotic decay then one is lead to the transport equations

$$2\nabla_x \phi^+ \cdot \nabla_x b_k - F\frac{\partial b_k}{\partial \xi_1} + ipb_k = i\Delta_x b_{k-1} \tag{3.8}$$

where $b_{-1} = 0$ and $k \geq 0$ and p was defined in (2.2). Solutions can be constructed by integrating along the characteristic curves $(x(t), \xi(t))$:

$$\begin{aligned} x'(t) &= 2\nabla_x \phi^+(x(t), \xi(t)) \\ \xi'(t) &= -F\mathbf{e}_1 \end{aligned}$$

where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbf{R}^n$. For initial conditions $\eta = \xi(0)$ and $y = x(0)$, $\xi(t) = \eta - Ft\mathbf{e}_1$ and

$$x(t) = y + 2\eta t - Ft^2\mathbf{e}_1 + 2\int_0^t \nabla_x \theta^+(x(s), \xi(s))\, ds \tag{3.9}$$

where $\theta^+(x, \xi) = \phi^+(x, \xi) - x \cdot \xi$. The notation $x(t, y, \eta) = x(t)$ and $\xi(t, \eta) = \xi(t)$ will be used when reference to the initial conditions is to be emphasized. To assure that the solutions of the transport equations can be extended to all of $\Gamma(\gamma, R, K, \kappa)$ requires checking that the characteristic curves that begin in this region stay within the domain of ϕ^+. More precisely, what is needed is:

Claim: *Given γ_0, $0 < \gamma_0 < \gamma_\phi$ there is $c_0 > 0$ so that if $(y, \eta) \in \Gamma(\gamma, R, K, \kappa)$, for some γ, $\gamma_0 < \gamma < \gamma_\phi$, $\kappa \geq 0$, $R > R_\phi$ and some $K > K_\phi$ then $(x(t, y, \eta), \xi(t, \eta))$ exists and belongs to the slightly larger region $\Gamma(\gamma, R', K', \kappa)$ for all $t \geq 0$ provided $R - R' > c_0$ and $K - K' > c_0$.*

The Lemma below provides an *a priori* estimate of the integral expression in the above expansion of $x(t)$; it is an analogue to Lemma 2.2 which is needed to establish the Claim

Lemma 2 *For all $R > R_\phi$ sufficiently large and γ, $0 < \gamma \leq \gamma_\phi$, K, $0 < K < R$ and $\kappa \geq 0$, and for all real $j, k \geq 0$ such that $k+j > 1/2$ there is a constant $C = C(j, k)$ so that, for (y, η) in $\Gamma(\gamma, R, K, \kappa)$, and $x(t) = x(t, y, \eta)$; $\xi(t) = \xi(t, \eta))$*

$$\begin{aligned} &\int_0^\infty |\Re x_1(t)|^{-k}(|\Re x_1(t)|/2 + (\Re \xi_1(t))^2/2F)^{-j}\, dt \\ &\leq \begin{cases} C|\Re y_1|^{-k+1}(|\Re y_1|/2 + (\Re \eta_1)^2/2F)^{-j-1/2} & \text{if } k > 1 \\ C|\Re y_1|^{-k}(|\Re y_1|/2 + (\Re \eta_1)^2/2F)^{-j+1/2} & \text{if } k \leq 1 \end{cases} \end{aligned} \tag{3.10}$$

A proof is included in the Appendix.

Proof of the Claim. The integral expression in the expansion (3.9) of $x(t)$ can be estimated as: for some $C_0 > 0$

$$|\int_0^t \nabla_x \theta^+(x(s), \xi(s))\, ds| < C_0 \langle R \rangle^{-\epsilon_\phi}.$$

according to the preceding Lemma in view of the estimate (2.7). Therefore

$$\begin{aligned}
|\Im x(t)| &\leq |\Im y| + 2|\Im \eta| t + C_0 R^{-\epsilon_\phi} \\
&\leq -\gamma \rho_V (\Re y_1 + K) - 2\gamma^2 \rho_V \Re \eta_1 t + C_0 R^{-\epsilon_\phi} \\
&\leq -\gamma \rho_V (\Re x_1(t) + K) - 2\rho_V(\gamma - \gamma^2)(-\Re \eta_1) t + C_0(1 + 2\gamma \rho_V) R^{-\epsilon_\phi} \\
&\leq -\gamma \rho_V (\Re x_1(t) + K')
\end{aligned}$$

provided K' and c_0 are chosen so that $K - K' > c_0 \geq C_0(2 + 1/\gamma \rho_V) R^{-\epsilon_\phi}$. A similar argument shows $\Re x_1(t) < R$ provided $\Re \eta_1 \leq 0$ and $R' > R$ is chosen large enough. Finally the inequalities $|\Im \xi(t)| \leq -\gamma^2 \rho_V \Re \xi_1(t)$ and $\Re \xi_1(t) < -\kappa \leq 0$ are elementary and do not depend on the choice of parameters. These estimates do not depend on $t \geq 0$ so that as long as the characteristics $(x(t), \xi(t))$ exist they stay within the domain of ϕ^+ and therefore can be extended by local existence theory to all $t \geq 0$. This verifies the Claim. □

Equation (3.8) can be solved when $k = 0$ there:

$$b_0(y, \eta) = \exp\left(-\int_0^\infty p(x(t, y, \eta), \xi(t, \eta))\, dt \right)$$

Then b_0 is holomorphic in a truncated cone $\Gamma(\gamma, R, K, \kappa)$ by the Claim and

$$b_0(y, \eta) - 1 = O(|\Re y_1|^{-\epsilon_\phi} [|\Re y_1|/2 + (\Re \eta_1)^2/2F]^{-1})$$

by (2.8) and Lemma 2. Define u_k by $b_k = b_0 u_k$ and substitute into the transport equations (3.8): for $k \geq 1$

$$\begin{aligned}
2\nabla_y \phi \cdot \nabla_y u_k - F \frac{\partial u_k}{\partial \eta_1} &= \frac{i}{b_0}[(\Delta_y b_0) u_{k-1} + 2\nabla_y b_0 \cdot \nabla_y u_{k-1} + b_0 \Delta_y u_{k-1}] \\
&= \sum_{|\alpha| \leq 2} \langle y_1 \rangle^{|\alpha| - 2} f_\alpha D_x^\alpha u_{k-1} \qquad (3.11)
\end{aligned}$$

for appropriate $f_\alpha(y, \eta)$ which are bounded and holomorphic on $\Gamma(\gamma, R, K, \kappa)$, and do not depend on k. Here the notation $\langle y_1 \rangle$ is extended to allow $y_1 \in \mathbf{C}$:

$\langle y_1\rangle = (1+y_1^2)^{1/2}$ is holomorphic on $\Gamma(\gamma, R, K, \kappa)$ and moreover $|\langle y_1\rangle|/\langle \Re y_1\rangle$ is bounded above and below by positive constants there. (The square root is chosen to be continuous on all but the negative real axis and so that $\sqrt{1} = 1$.)

It is further possible to show that $b_0(x,\xi) - 1$ converges to zero in the "perpendicular" directions, that is as $|(x_\perp, \xi_\perp)|$ gets large. This follows from the comparable property for p established in Proposition 2.1 by way of the following Lemma.

Lemma 3 *Let $\Gamma(\gamma, R, K, \kappa) \subseteq \Gamma(\gamma, R', K', \kappa)$ be as in the* **Claim** *above. Suppose that $q(x,\xi)$ is continuous on $\Gamma(\gamma, R', K', \kappa)$ and for some $k > 1$ and $j \geq 0$*

$$\begin{aligned} |\Re x_1|^k ((\Re\xi_1)^2/2F + |\Re x_1|/2)^j |q(x,\xi)| &\leq C \\ \lim_{|(x,\xi)|\to\infty} |\Re x_1|^k ((\Re\xi_1)^2/2F + |\Re x_1|/2)^j |q(x,\xi)| &= 0 \end{aligned}$$

for some constant $C > 0$. Then, for any $j' < j$

$$\lim_{|(y,\eta)|\to\infty} |\Re y_1|^{k-1} ((\Re\eta_1)^2/2F + |\Re y_1|/2)^{j'+1/2} \int_0^\infty |q(x(t,y,\eta), \xi(t,\eta))|\, dt = 0$$

where the limit is taken inside $\Gamma(\gamma, R, K, \kappa)$.

A proof is given in the Appendix.

Remark: Applying this result with $q = p$ it follows that b_0 satisfies (3.6) with b there replaced by b_0.

Solving (3.11) involves constructing Banach spaces which use the following function m_1 as a weight function:

$$m_1(x) = -\Re x_1 - \frac{1}{\gamma^* \rho_V}(\langle \Im x\rangle + R^*)$$

for positive constants $0 < \gamma^* < \gamma_\phi$, $R^* > R_\phi$ to be specified below. The solution will be constructed on $\Omega \subseteq \mathbf{C}^{2n}$ where

$$\begin{aligned} \Omega_1 &= \{x \in \mathbf{C}^n : m_1(x) > 0\} \\ \Omega &= \Omega_1 \times \{\xi \in \mathbf{C}^n : |\Im \xi| < -(\gamma^*)^2 \rho_V \Re\xi_1, \Re\xi_1 < -\kappa\} \end{aligned}$$

$0 < \kappa < \kappa^*$ By an elementary calculation, $\Omega \subset \Gamma(\gamma^*, R, R, \kappa)$ where $R = R^*/(\rho_V \gamma^*)$ so that Ω is within the domain of ϕ provided R^* is large enough.

Record the properties of m_1.

Lemma 4 *In general*

$$m_1(x) \leq (1 + (\gamma^* \rho_V)^{-2})^{1/2} dist(x, \partial\Omega_1)$$

where $dist(x, \partial\Omega_1)$ *denotes the distance from* x *to the boundary* $\partial\Omega_1$ *of* Ω_1. *Also* $m_1(x) \leq -\Re x_1$. *Fix* δ, $0 < \delta < 1 - \gamma^*$. *Then, provided* R^* *is large enough (depending on* δ*), whenever* $\langle \Im x \rangle \leq -\rho_V (\gamma^*)^2 \Re x_1 - R^*$

$$(1 - \gamma^*)(-\Re x_1) \leq \quad m_1(x) \quad \leq -\Re x_1 \tag{3.12}$$

$$(1 - \gamma^*) dist(x, \partial\Omega_1) \leq \quad m_1(x) \quad \leq (1 + (\gamma^* \rho_V)^{-2})^{1/2} dist(x, \partial\Omega_1) \tag{3.13}$$

and for any $(y, \eta) \in \Omega$

$$\frac{d}{dt} m_1(x(t, y, \eta)) \geq 2Ft - 2\delta \Re \eta_1. \tag{3.14}$$

Proof. Clearly $m_1(x) \leq -\Re x_1$. Therefore the inequality (3.12) follows directly because $\langle \Im x \rangle \leq -\rho_V (\gamma^*)^2 \Re x_1 - R^*$. Consider next (3.13). Since $m_1(y) = 0$ whenever $y \in \partial\Omega$, it follows that $m_1(x) \leq |\nabla m_1| \text{dist}(x, \partial\Omega)$. But $|\nabla m_1(x)|^2 \leq 1 + (\gamma^* \rho_V)^{-2}$ which implies the upper bound for m_1 in (3.13). (m_1 is regarded as a function of $2n$ real variables here.) Now suppose that $|\Im x| \leq -\rho \gamma^* \Re x_1 - R^*$. It will be shown that $\text{dist}(x, \partial\Omega) \leq |\Re x_1|$ and this will imply the lower bound for m_1 of (3.13) in view of (3.12). Consider therefore $y = x - \Re x_1 \mathbf{e}_1$; then $m_1(y) < 0$ so that $\text{dist}(x, \partial\Omega) \leq |x - y| = |\Re x_1|$ which proves (3.13).

From the expansion (3.9) of $x(t)$

$$\begin{aligned}
\frac{d}{dt} m_1(x(t)) &= -\frac{d}{dt} \Re x_1(t) - \frac{1}{\rho_V \gamma^*} \frac{\Im x(t)}{\langle \Im x(t) \rangle} \cdot \frac{d}{dt} \Im x(t) \\
&\geq -2\Re \eta_1 + 2Ft - 2\left|\frac{\partial}{\partial x_1} \Re \theta^+(x(t), \xi(t))\right| \\
&\quad - \frac{2}{\rho_V \gamma^*} (|\Im \eta| + |\nabla_x \Im \theta^+(x(t), \xi(t))|) \\
&\geq 2Ft + 2(1 - \gamma^*)(-\Re \eta_1) - C|\Re x_1(t)|^{-1/2-\epsilon}
\end{aligned}$$

by the estimate for $\nabla_x \theta^+ = \nabla_x(\phi^+ - \xi)$ of Proposition 2.1. This implies (3.14) because $|\Re x_1(t)|$ will be large for all $t \geq 0$, provided R^* is chosen large depending on δ (by the earlier "Claim") and because $-\Re \eta_1 > \kappa$. $\square$.

Introduce a space $\mathcal{A}(\Omega)$ of formal series $u = \sum_{k\geq 0} u_k(x,\xi)$ in which to construct the symbol. There will be two weights, $m_1(x)$ and

$$m_2(x,\xi_1) = \delta(\Re\xi_1)^2/2F + m_1(x)/2$$

where $\delta > 0$ is the constant of the preceding Lemma. Define $\mathcal{A}(\Omega)$ to be the set of all formal series $u = \sum_{k\geq 0} u_k(x,\xi)$ where each $u_k(x,\xi)$ is holomorphic on Ω, and

$$(k+1)^{-3k/2} m_1(x)^k m_2(x,\xi_1)^{k/2} |u_k(x,\xi)| \leq C(u,k) \qquad (3.15)$$

$$\lim_{(x,\xi)\to\infty} m_1(x)^k m_2(x,\xi_1)^{k/2} |u_k(x,\xi)| = 0 \text{ if } k \geq 1 \qquad (3.16)$$

(in Ω) for some constant $C(u,k)$; and moreover $u_0(x,\xi) = u_0$ is a constant. If $C(u,k)$ is the minimal constant in (3.16) and if there is $\mu > 0$ so that

$$\|u\|_\mu = \sum_{k\geq 0} C(u,k)\mu^k < \infty$$

then u is said to belong to $\mathcal{A}_\mu(\Omega)$. Then $\mathcal{A}_\mu(\Omega)$ is a Banach space. Define also $\mathcal{A}(\Omega) = \cup_{\mu>0}\mathcal{A}_\mu(\Omega)$.

The next Lemma will serve to keep track of the asymptotic behavior of the solution of the transport equations (3.11).

Lemma 5 *If ν is any positive integer then there is a constant $C_\nu > 0$ so that for any $u \in \mathcal{A}(\Omega)$ and any multi-index $\alpha \in \mathbf{N}_0^n$, $0 \leq |\alpha| \leq \nu$, and any $k \geq 0$,*

$$(k+1)^{-(3k/2)-|\alpha|} m_1(x)^{k+\nu} m_2(x,\xi_1)^{k/2} |\langle x_1\rangle^{-\nu+|\alpha|} D_x^\alpha u_k(x,\xi)| \leq C_\nu C(u,k)$$

$$\lim_{|(x,\xi)|\to\infty} m_1(x)^{k+\nu} m_2(x,\xi_1)^{k/2} |\langle x_1\rangle^{-\nu+|\alpha|} D_x^\alpha u_k(x,\xi)| = 0, \text{ if } k \geq 1.$$

Proof. Suppose first that $\alpha = 0$. On Ω, $|\langle x_1\rangle| \geq C|\Re x_1| \geq Cm_1(x)$ by (3.12) so that

$$|\langle x_1\rangle^{-\nu} u_k(x,\xi)| \leq \frac{C(u,k)(k+1)^{3k/2}}{C^\nu m_1(x)^{k+\nu} m_2(x,\xi_1)^{k/2}}$$

which is the required bound provided $C_\nu \geq 1/C^\nu$. The conclusion about the limiting behavior as $|(x,\xi)| \to \infty$ in the case $\alpha = 0$ follows similarly.

Next suppose $|\alpha| = \nu$; the general case follows from the special case by adapting the initial argument. To avoid introducing more notation, assume also that all the ν derivatives are in one variable, x_j say, $1 \leq j \leq n$; that is an unmixed partial derivative. The case of arbitrary $\alpha \in \mathbf{N}_0^n$ is similar. By Cauchy's formula,

$$\frac{\partial^\nu u_k}{\partial x_j^\nu}(x,\xi) = \frac{\nu!}{2\pi i}\int_{|y_j - x_j| = \sigma m_1(x)} \frac{u_k(y)\,dy_j}{(y_j - x_j)^{\nu+1}}$$

provided that $\sigma > 0$ is chosen so small that $\sigma m_1(x) < dist(x, \partial\Omega)$; σ can be chosen independent of x by (3.13). Therefore

$$|\frac{\partial^\nu u_k}{\partial x_j^\nu}(x,\xi)| \leq \frac{\nu!}{2\pi}\int_{|y_j - x_j| = \sigma m_1(x)} \frac{C(u,k)(k+1)^{3k/2} dy_j}{m_1(y)^k m_2(y,\xi_1)^{k/2}\sigma^{\nu+1} m_1(x)^{\nu+1}}$$

On the circle, $|y_j - x_j| = \sigma m_1(x)$, $m_1(y) \geq (1 - C\sigma)m_1(x)$ because $|\nabla m_1(x)|$ is bounded by some constant C. Therefore

$$|\frac{\partial^\nu u_k}{\partial x_j^\nu}(x,\xi)| \leq \frac{\nu! C(u,k)(k+1)^{3k/2}}{\sigma^\nu (1 - C\sigma)^{3k/2} m_1(x)^{k+\nu} m_2(x,\xi_1)^{k/2}}.$$

The choice $\sigma = 2\nu/(C(3k+2\nu))$ minimizes the right side of the above inequality and gives the desired bound on the derivative of u_k. Similar reasoning shows that the derivative converges to 0 as $|(x,\xi)| \to \infty$; one needs only note that $|y| \to \infty$ as $|x| \to \infty$. This completes the proof. □

The preceding Lemma will be instrumental for controlling the asymptotic behavior of the non-homogeneous term in (3.11). It will be further necessary to describe the asymptotic behavior of certain solutions of

$$2\nabla_x \phi^+ \cdot \nabla_x u_k - F\frac{\partial u_k}{\partial \xi_1} = w$$

for a given function w defined on Ω. (Fix k for the moment.) The solution can be obtained by integrating along the characteristics $(x(t), \xi(t))$. Observe that the characteristics that begin in Ω stay in Ω by (3.14). Introduce therefore the notation

$$(2\nabla_x \phi^+ \cdot \nabla_x - F\frac{\partial}{\partial \xi_1})^{-1} w\,(y,\eta) = \int_0^\infty w(x(t,y,\eta), \xi(t,\eta))\,dt \qquad (3.17)$$

for a solution evaluated at (y,η). Assume that $w(x,\xi)$ is holomorphic in Ω and

$$|w(x,\xi)| < Cm_1(x)^{-k}m_2(x,\xi_1)^{-j}$$

for some $k > 1$ and $j \geq 0$. By (3.14)

$$\begin{aligned}
&|(2\nabla_x\phi^+ \cdot \nabla_x - F\frac{\partial}{\partial\xi_1})^{-1}w\ (y,\eta)| \\
&\leq\ C\int_0^\infty (m_1(y) - 2\delta\Re\eta_1 t + Ft^2)^{-k} \times \\
&\qquad (\delta\frac{(\Re\eta_1)^2}{2F} + \frac{m_1(y)}{2} - 2\delta\Re\eta_1 t + F(1+\delta)t^2/2)^{-j}\,dt \\
&\leq\ C\int_0^\infty (m_1(y) - 2\delta\Re\eta_1 t + F\delta t^2)^{-k} \times \\
&\qquad (\delta\frac{(\Re\eta_1)^2}{2F} + \frac{m_1(y)}{2} - 2\delta\Re\eta_1 t + F\delta t^2)^{-j}\,dt
\end{aligned}$$

since $0 < \delta < 1$. Lemma 2.2 applies: for $k > 1$

$$|(2\nabla_x\phi^+ \cdot \nabla_x - F\frac{\partial}{\partial\xi_1})^{-1}w\ (y,\eta)| \leq \frac{CC_0}{\sqrt{F\delta k}}m_1(y)^{-k+1}m_2(y,\eta_1)^{-j-1/2}. \tag{3.18}$$

In addition it is further necessary to know that if

$$\lim_{|(x,\xi)|\to\infty} m_1(x)^k m_2(x,\xi)^j |w(x,\xi)| = 0$$

then

$$\lim_{|(x,\xi)|\to\infty} m_1(y)^{k-1}m_2(y,\eta_1)^{j+1/2}|(2\nabla_x\phi \cdot \nabla_x - F\frac{\partial}{\partial\xi_1})^{-1}w\ (y,\eta)| = 0 \tag{3.19}$$

The argument of the preceding paragraph applies here because the added assumption implies that the constant C of that argument can chosen arbitrarily small provided $|(y,\eta)|$ is adequately large. It is only necessary to check that $|(y,\eta)| \to \infty$ (in Ω) assures that the characteristic path $|(x(t,y,\eta),\xi(t,\eta)| \to \infty$ uniformly in $t \geq 0$. This latter fact follows from (3.9) by way of (3.10).

It is convenient to rewrite the transport equations (3.11) as $u = 1 + Tu$ where T is defined on $\mathcal{A}(\Omega)$ by

$$(Tu)_k = (2\nabla_x\phi^+ \cdot \nabla_x - F\frac{\partial}{\partial\xi_1})^{-1}\sum_{|\alpha|\leq 2}\langle y_1\rangle^{|\alpha|-2}f_\alpha D_x^\alpha u_{k-1}$$

for $k \geq 1$. Setting $v = u - 1$, (3.11) becomes

$$v - Tv = T1$$

By (3.18) and Lemma 5, there is a constant $C > 0$ not depending on k or v so that

$$|(Tv)_k(y,\eta)| \leq \frac{CC(v,k-1)k^{3(k-1)/2+2}}{(k+1)^{1/2}m_1(y)^k m_2(y,\eta_1)^{k/2}}$$

for any v in $\mathcal{A}(\Omega)$. This estimate and (3.19) assure that Tv belongs to $\mathcal{A}(\Omega)$ if v does. (There is one special case: checking (3.16) for $(Tv)_1$ or equivalently for $\langle y_1\rangle^{-2} f_0 = i\Delta_y b_0/b_0$; see the remark after Lemma 3.) Therefore the preceding estimate on $(Tv)_k(y,\eta)$ implies that as an operator on $\mathcal{A}_\mu(\Omega)$

$$\|T\| \leq C\mu < 1$$

for μ small enough. This assures that $I - T$ is invertible and guarantees a solution $v \in \mathcal{A}_\mu(\Omega)$ of $(I-T)v = T1$, where I denotes the identity operator. Then $u = 1+v$ and this completes the construction of the sequence $b_k = b_0 u_k$, $k \geq 0$. Since u defines an element of $\mathcal{A}_\mu(\Omega)$, there is a constant $C > 0$ so that

$$\begin{aligned} |b_k(x,\xi)| &\leq \frac{C(b,k)(k+1)^{3k/2}}{m_1(x)^k(\delta(\Re\xi_1)^2/2F + m_1(x)/2)^{k/2}} \\ &\leq \frac{C^{k+1}(k+1)^{3k/2}}{(-\Re x_1)^k((\Re\xi_1)^2/2F - \Re x_1/2)^{k/2}} \end{aligned}$$

on the subset of Ω where $-\Re x_1 \geq (1/\rho_V(\gamma^*)^2)(|\Im x| + R^*)$ by Lemma 4. Thus each b_k is defined on $\Gamma((\gamma^*)^2, R, R, \kappa)$ where $R = R^*/\rho_V(\gamma^*)^2$.

To define the symbol b, it remains to sum the b_k in a convergent series. Introduce therefore $\tilde{\chi}_{(1,\infty)} \in C^\infty(0,\infty)$ chosen to be increasing and so that $\tilde{\chi}_{(1,\infty)}(s) = 0$ if $s < 1$ and $\tilde{\chi}_{(1,\infty)}(s) = 1$ if $s > 2$. Define

$$b(x,\xi) = \sum_{0 \leq k \leq k_\sigma} b_k(x,\xi) + \sum_{k > k_\sigma} b_k(x,\xi)\tilde{\chi}_{(1,\infty)}\left(\frac{\Re x_1}{C_1(k+1)}\right)$$

where k_σ is the arbitrary nonnegative integer in the statement of Proposition 1 and where C_1 is some constant to be chosen. The sum is locally finite and therefore defines a function holomorphic in $(x_\perp, \xi)$.

Verify (3.3). The case when $j = 0 = k$ is illustrative. Substituting the above defining expansion for b into the definition of t^+ and taking into account the transport equations leads to the estimate

$$|t^+(x,\xi)| < C_0(k_\sigma)\langle \Re\xi_1\rangle^{-k_\sigma}\left[1 - \tilde{\chi}_{(1,\infty)}\left(\frac{\Re x_1}{C_1(k_\sigma+1)}\right) + \sum\left(\frac{C_2}{C_1}\right)^{3k/2}\right]$$

for some positive constants $C_0(k_\sigma)$ and C_2 and where the sum extends over $k > c\langle\Re x_1\rangle$, again for a positive constant c. The verification of the above bound involves the estimates on b_k already noted and comparable estimates for the first two derivatives of b_k in x_1 which follow from Lemma 5 Recall also the bounds on the x_1 derivative of ϕ^+ from Proposition 1; the elementary inequality $(k+1)^{k_\sigma} < 2^{kk_\sigma}$ is used as well. The estimate (3.3) in the case $j = 0 = k$ follows directly by choosing $C_1 > C_2$. The verification of (3.7) is very similar; this time the decay in the $\Re\xi_1$ variable is disregarded as uninteresting for the present purposes.

The estimates (3.5) and (3.6) for $b-1$ can now be verified. Since b_0 satisfies (3.5) and (3.6) (replace b there by b_0) it suffices to check the estimate for

$$\tilde{v}(x,\xi) = \sum_{1\le k\le k_\sigma} v_k(x,\xi) + \sum_{k>k_\sigma} v_k(x,\xi)\tilde{\chi}_{(1,\infty)}\left(\frac{\Re x_1}{C_1(k+1)}\right)$$

(v_k as defined above) because $b - 1 = (b_0-1)\tilde{v} + (b_0-1) + \tilde{v}$. Because v is in $\mathcal{A}_\mu$ and $v_0 = 0$

$$|\tilde{v}(x,\xi)| < Cm_1(x)^{-1}m_2(x,\xi_1)^{-1} < C'(\Re x_1)^{-1}((\Re\xi_1)^2/2F + m_1(x)/2)^{-1/2}$$

on $\Gamma((\gamma^*)^2, R, R, \kappa)$ where $R = R^*/\rho_V(\gamma^*)^2$. ($C_1$ may need to be increased.) By Lemma 5 the same argument applies to the derivatives in x_1 (without further increasing C_1). This completes the proof of Proposition 1. □

4 Analyticity of the Symbol.

Equation (3.7) says that the symbol b constructed in Proposition 3.1 is almost analytic in the x_1 in the sense that will be made precise by calling on a result of P. Laubin [27, Theorem 3.2]. To state the special case of that result needed below, some notation is needed. For any $\Omega \subseteq \mathbf{R}^{2n-1}$ and $\delta > 0$, define the cone $\Gamma_{\Omega,\delta} \subseteq \mathbf{C}^{2n}$

$$\Gamma_{\Omega,\delta} = \{(x,\xi) : (\Re x_\perp, \Re \xi) \in \Omega, |\Im x_1| \leq \delta \langle \Re x_1 + K_\sigma \rangle, |\Im x_\perp|^2 + |\Im \xi|^2 < \delta^2\}$$

where K_σ is the fixed constant of Proposition 3.1. Suppose now that $\Omega \subseteq \mathbf{R}^{2n-1}$ is open. Then a C^∞ function f is said to be *exponentially decreasing* if for every compact subset $\mathcal{K}$ of Ω there is $\delta, \mu, C > 0$ so that $f(x,\xi)$ is defined for all $(x,\xi) \in \Gamma_{\mathcal{K},\delta}$ and

$$|f(x,\xi)| < Ce^{-\mu|x_1|} \tag{4.1}$$

Theorem 1 (Laubin) *For any $f \in C^\infty(\Gamma_{\Omega,\delta_0})$ such that, for $1 \leq j \leq n$,*

$$\partial_{\overline{x}_j} f \text{ and } \partial_{\overline{\xi}_j} f \text{ are exponentially decreasing}$$

there is a C^∞ function g which is exponentially decreasing and for $1 \leq j \leq n$

$$\partial_{\overline{x}_j} f = \partial_{\overline{x}_j} g \quad \text{and} \quad \partial_{\overline{\xi}_j} f = \partial_{\overline{\xi}_j} g.$$

For an (admirably clear) proof see [27, Theorem 3.2]. What will be required below is a "uniform" version of this result. Define $f \in C^\infty(\Gamma_{\Omega,\delta_0})$ to be *uniformly exponentially decreasing* on Ω if there is $\delta, \mu, C > 0$ so that (4.1) holds on all of $\Gamma_{\Omega,\delta}$. The intended application is to $f = b$ or more correctly to b appropriately extended so that $b(x,\xi) = 1$ if $\Re x_1 > -R_\sigma$: explicitly extend b as

$$1 + \tilde{\chi}_{(-\infty,-2)}(\Re x_1/R)[b(x,\xi) - 1] \tag{4.2}$$

where

$$\tilde{\chi}_{(-\infty,-2)}(x_1) = \begin{cases} 1 & \text{if } x_1 < -5/2 \\ 0 & \text{if } x_1 > -3/2 \end{cases} \tag{4.3}$$

and where $R > R_\sigma$ is a parameter to be chosen later. The extended function will also be called b; its domain contains $\Gamma_{\Omega,\delta}$ where

$$\Omega \equiv \{(x_\perp, \xi) \in \mathbf{R}^{2n-1} : \xi_1 < -\kappa_\sigma\}$$

and $0 < \delta < \gamma_\sigma \rho_V$. Since $\partial_{\overline{x}_j} b$ and $\partial_{\overline{\xi}_j} b$ are uniformly exponentially decreasing Laubin's Theorem 1 applies. Let g be the function whose existence is guaranteed by that result. Claim that not only is g exponentially decreasing but in fact it is uniformly exponentially decreasing on the smaller set

$$\Omega' = \{(x_\perp, \xi) \in \mathbf{R}^{2n-1} : \xi_1 < -3\kappa_\sigma/2\}.$$

To verify the claim, suppose that $\mathcal{K} \subseteq \Omega$ is a compact set, $\delta, \mu, C > 0$ are the constants so that g satisfies (4.1) (replace f there by g) on $\Gamma_{\mathcal{K},\delta}$. To establish the claim, it will be shown that if $\mathcal{K}_{\mathbf{d}} \subseteq \Omega$ is any translate $\mathcal{K}_{\mathbf{d}} = \mathcal{K} + \mathbf{d}$ of $\mathcal{K}$ by a vector in $\mathbf{d} \in \mathbf{R}^{2n}$ that is perpendicular to the x_1 direction ($d_1 = 0$) and if $\mathcal{K}_{\mathbf{d}}$ is at least as far from the boundary of Ω as $\mathcal{K}$ ($d_{n+1} \leq 0$) then b satisfies (4.1) on $\Gamma_{\mathcal{K}_{\mathbf{d}},\delta}$ with the same positive constants C, μ, δ. For let $b_{\mathbf{d}}(x, \xi) = b((x, \xi) + \mathbf{d})$. Because of the uniformity of the exponential decrease of $\partial_{\overline{x}_j} b$ and $\partial_{\overline{\xi}_j} b$, the derivatives $\partial_{\overline{x}_j} b_{\mathbf{d}}$ and $\partial_{\overline{\xi}_j} b_{\mathbf{d}}$ satisfy exactly the same bounds on all of Ω, as do $\partial_{\overline{x}_j} b$ and $\partial_{\overline{\xi}_j} b$ $1 \leq j \leq n$. (Since $d_{n+1} \leq 0$, $b_{\mathbf{d}}$ is defined on all Ω.) The bounds for the function g of Laubin's theorem depend only on these bounds and $\mathcal{K}$ and not on the particular function f or even on the choice of g (which may not be unique) provided g is as in [15, Lemma 4.4.1] which is the existence result Laubin bases his Theorem on. (This independence is made explicit in equation (4.4.2) of [15].) It follows that the function whose existence is guaranteed by Laubin's result applied to $b_{\mathbf{d}}$ can be chosen as the translate $g_{\mathbf{d}}(x, \xi) \equiv g((x, \xi) + \mathbf{d})$ and $g_{\mathbf{d}}$ satisfies the same estimates as g on $\mathcal{K}$ and this completes the proof of the claim. For future reference we shall denote g as b_e and write

$$b = b_e + b_{\mathcal{A}}$$

so that b_e is uniformly exponentially decreasing on $\Gamma_{\Omega',\delta}$, for some $\delta > 0$ and $b_{\mathcal{A}}$ is analytic there.

The phase ϕ and symbol a of J are defined by cutting off and extending ϕ^+ and b appropriately. Let

$$a^+(x, \xi) = 1 + \tilde{\chi}_{(-\infty,-2)}(\Re\xi_1/\kappa_\sigma)(b(x, \xi) - 1)$$

for $\tilde{\chi}_{(-\infty,-2)}$ as in (4.3). Define the incoming analogues of a^+ and ϕ^+ by "time reversal":

$$a^-(x, \xi) = \overline{a^+(\overline{x}, -\overline{\xi})} \qquad \phi^-(x, \xi) = -\phi^+(x, -\xi). \tag{4.4}$$

(Recall ϕ^+ is real valued on $\mathbf{R}^{2n}$.) It will also be convenient to glue the incoming and outgoing symbol and phase as follows:

$$a(x,\xi) = a^+(x,\xi) + a^-(x,\xi) - 1 = a^+(x,\xi)a^-(x,\xi) \tag{4.5}$$

$$\phi(x,\xi) = \phi^+(x,\xi)\tilde{\chi}_{(-\infty,0)}(\Re\xi_1/\kappa_\sigma) + \phi^-(x,\xi)\tilde{\chi}_{(0,\infty)}(\Re\xi_1/\kappa_\sigma) \tag{4.6}$$

($\tilde{\chi}_{(-\infty,0)}$ and $\tilde{\chi}_{(0,\infty)}$ as in (1.3)). Corresponding to the breakdown $b = b_{\mathcal{A}} + b_e$, there is $a = a_{\mathcal{A}} + a_e$ where

$$\begin{aligned} a_e(x,\xi) &= \tilde{\chi}_{(-\infty,-2)}(\Re\xi_1/\kappa_\sigma)b_e(x,\xi) + \tilde{\chi}_{(-\infty,-2)}(-\Re\xi_1/\kappa_\sigma)\overline{b_e(\overline{x},-\overline{\xi})} \\ a_{\mathcal{A}}(x,\xi) &= 1 + \tilde{\chi}_{(-\infty,-2)}(\Re\xi_1/\kappa_\sigma)(b_{\mathcal{A}}(x,\xi) - 1) \\ &\quad + \tilde{\chi}_{(-\infty,-2)}(-\Re\xi_1/\kappa_\sigma)(\overline{b_{\mathcal{A}}(\overline{x},-\overline{\xi})} - 1) \end{aligned}$$

Corollary 2 *For $b = b_{\mathcal{A}} + b_e$ and $a = a_{\mathcal{A}} + a_e$ as defined above and for any $j,k \in \mathbf{N}_0$, the derivatives $\partial^j_{\overline{x}_1}\partial^k_{x_1}b_e$ (resp. $\partial^j_{\overline{x}_1}\partial^k_{x_1}a_e$) are uniformly exponentially decreasing on Ω' (resp. on $\mathbf{R}^{2n-1}$). Further there is $\delta > 0$ ($\delta = \delta(M)$) so that $b_{\mathcal{A}}$ is analytic (in $2n$ variables) on $\Gamma_{\Omega',\delta}$ whereas $a_{\mathcal{A}} \in C^\infty(\Gamma_{\mathbf{R}^{2n-1},\delta})$ is analytic on $\Gamma_{\mathbf{R}^{2n-1},\delta} \cap \{|\Re\xi_1| > 5\kappa_\sigma/2\}$ and, for fixed ξ_1, $|\Re\xi_1| < 5\kappa_\sigma/2$ is analytic in $(x,\xi_\perp)$ on the cross section of $\Gamma_{\mathbf{R}^{2n-1},\delta}$.*

Proof: The conclusions, as far as a is concerned, follow directly from those for b and the definitions. Consider therefore b. The decomposition $b = b_{\mathcal{A}} + b_e$ has already been constructed in the preceding discussion; it remains only to discuss the derivatives of b_e. Only first order derivatives are considered. The decomposition applies equally well to the derivatives of b as to b:

$$\partial b/\partial\Re x_1 = h + e_1 \qquad \text{and} \qquad \partial b/\partial\Im x_1 = ih + e_2$$

where e_1 and e_2 are uniformly exponentially decreasing and h is analytic on $\Gamma_{\Omega',\delta}$. (The analytic terms h and ih are related because of (3.7).) It is quite possible however that the decompositions of the derivatives are not the derivatives of the decomposition. It remains therefore to check that the derivatives of b_e in $\Re x_1$ and $\Im x_1$ are uniformly exponentially decreasing. Only the first order derivatives will be checked. Certainly when $x_1 > 0$ then $b \equiv 1$ so that b_e is both uniformly exponentially decreasing and analytic. It follows from Cauchy's integral formula (and it's derivatives) that the derivatives of b_e are exponentially decaying in $x_1 > 0$. Next suppose $x_1 < 0$. Express b as

$$b = \int (h + e_1)d\Re x_1 + (ih + e_2)d\Im x_1$$

where the path integral is over a path from $\Re x_1 = -\infty$, $\Im x_1 = 0$. The decay of the gradient of b assures the integral is well defined. (See Proposition 3.1.) The path integral gives an alternative decomposition of b as $\tilde{b}_{\mathcal{A}} + \tilde{b}_e$ where $\tilde{b}_{\mathcal{A}}$ (corresponding to h) is analytic and $\tilde{b}_e$ is exponentially decaying in $\Re x_1 < 0$. The difference $\tilde{b}_e - b_e$ is both exponentially decaying in $x_1 < 0$ and analytic and so, as before, the derivatives are also exponentially decaying. Thus the first derivatives of b_e must themselves be exponentially decaying. This verifies the result for first order derivatives and the general case is similar. □

5 Continuation of Integral Operators.

In this section the operator $T_0(\lambda)$ of restriction to the manifold of constant free energy λ is introduced. Then a class of Fourier integral operators Q is introduced for which $QT_0(\lambda)^*$ can be continued analytically in λ. The class includes $Q = (H_{\mathcal{A}}J - JH_0)$ restricted to the outgoing states. This continuation result is a building block from which the continuation results for the resolvent and scattering matrix are derived in the next Sections; it is an analogue of Yajima's [34, Lemma 2.1].

As a preliminary to defining $T_0(\lambda)$, introduce a spectral representation U for $H_0 = -\Delta + Fx_1$. Define

$$G(\xi) = (1/3)\xi_1^3 + \xi_1(\xi_2^2 + \ldots + \xi_n^2) \tag{5.1}$$

and

$$Uv(x) = F^{-n/2} \int e^{i(x\cdot\xi - G(\xi))/F} \hat{v}(\xi)\, d_1\xi \tag{5.2}$$

for all v in $\mathcal{S}(\mathbf{R}^n)$ so that U extends to a unitary operator on $L^2(\mathbf{R}^n)$ also denoted U, and

$$H_0 = U^* x_1 U.$$

Define, for each real λ, $T_0(\lambda)$ as an operator on $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{n-1})$ by $T_0(\lambda)v(x_\perp) = Uv(\lambda, x_\perp)$, i.e. U followed by restriction to the plane $x_1 = \lambda$ so that

$$T_0(\lambda)v(x_\perp) = F^{-n/2} \int e^{i(\lambda\cdot\xi_1 + x_\perp\cdot\xi_\perp - G(\xi))/F} \hat{v}(\xi)\, d_1\xi$$

where $x_\perp = (x_2, \ldots, x_n)$ is the projection of x on its last $n-1$ coordinates. Of courses $T_0(\lambda)$ is not defined on all of $L^2(\mathbf{R}^n)$ but it is defined on $H^s(\mathbf{R}) \otimes L^2(\mathbf{R}^{n-1})$ provided $s > 1/2$. Here $H^s(\mathbf{R})$ denotes the usual

Sobolev space: $H^s(\mathbf{R}) = \{u \in L^2(\mathbf{R}) : (1 + D_1^2)^{s/2}u$ belongs to $L^2(\mathbf{R})\}$; where $D_1 = -i\partial/\partial x_1$. Abbreviate $H^s(\mathbf{R}) \otimes L^2(\mathbf{R}^{n-1})$ as $H^s \otimes L^2$. It can further be checked that $T_0(\lambda)$ is strongly continuous in λ on $H^s \otimes L^2$ and that the Fourier transform of $T_0(\lambda)^*u$ is simply,

$$[T_0(\lambda)^*u]\hat{\ }(\xi) = (2\pi)^{-1/2}F^{-n/2}e^{-i\lambda\xi_1/F+iG(\xi)/F}\hat{u}(\xi_\perp/F) \tag{5.3}$$

for $u \in \mathcal{S}(\mathbf{R}^{n-1})$ in the sense of tempered distributions for example.

One consequence of these definitions is the following observation about the spectral measure E_0 of H_0. Suppose u, v are in $H^s \otimes L^2$ for some $s > 1/2$ and let $d(E_0(\lambda)u, v)/d\lambda$ be the Radon Nikodym derivative of the measure $(E_0(\cdot)u, v)$ with respect to Lebesgue measure. Then

$$d(E_0(\lambda)u, v)/d\lambda = (T_0(\lambda)u, T_0(\lambda)v). \tag{5.4}$$

Many of the operators to be encountered below are pseudo-differential operators defined by

$$\Psi u(x) \;=\; \text{Os-}\iint e^{-i\xi\cdot y}\rho(x, x+y, \xi)u(x+y)\, dyd\xi \tag{5.5}$$

for all $u \in \mathcal{S}(\mathbf{R}^n)$. Here "Os-" indicates that the integrals are oscillatory integrals; see [24], for example. It will be convenient to have a criterion that assures that Ψ is bounded and that is given by the Calderón-Vaillancourt theorem [6], a special case of which will now be stated. For the present purposes, the symbol ρ will be restricted to $C^\infty(\mathbf{R}^{3n})$. Define for each $k = 0, 1, 2, \dots$

$$|\rho|_k \;\equiv\; \sup\{|D_x^\alpha D_y^\gamma D_\xi^\beta \rho(x, y, \xi)|, |\alpha+\beta+\gamma| \le k, (x, y, \xi) \in \mathbf{R}^{3n}\} \tag{5.6}$$

The version of the Calderón-Vaillancourt theorem [6] (or see [24, p. 224], for example) convenient for the present applications says that there is a constant $C > 0$ and integer $k = 0, 1, 2, \dots$ not depending on ρ so that

$$\|\Psi\| \;\le\; C|\rho|_k$$

where $\|\cdot\|$ denotes the operator norm on $L^2(\mathbf{R}^n)$.

One application is to show (as in [17, 22]) the boundedness of operators like J and $H_{\mathcal{A}}J - JH_0$. Introduce therefore a slightly more general class of integral operators Q that will be convenient for applications. Define

$$Qu(x) \;=\; \int e^{i\phi(x,\xi)}q(x, \xi)\hat{u}(\xi)\, d\xi \qquad \text{for } u \in \mathcal{S}(\mathbf{R}^n) \tag{5.7}$$

where the symbol $q(x,\xi)$ is any $C^\infty(\mathbf{R}^{2n})$ function and ϕ is the phase function of (4.6).

Suppose Q_1 and Q_2 are two such operators defined with symbols q_1 and q_2 in $C_b^\infty(\mathbf{R}^{2n})$. Then each Q_j^* maps $\mathcal{S}(\mathbf{R}^n)$ to itself, $j=1,2$, and for $u \in \mathcal{S}(\mathbf{R}^n)$

$$Q_1 Q_2^* u(x) = \iint e^{i(\phi(x,\xi)-\phi(y,\xi))} q_1(x,\xi)\overline{q_2(y,\xi)} u(y)\, dy\, d\xi.$$

Make a change of variable: $\xi' \mapsto \xi$ where

$$\xi' = \int_0^1 \nabla_x \phi(y+s(x-y),\xi)\, ds.$$

The mapping from ξ to ξ' has a global inverse, provided that the parameter $\delta > 0$ is chosen small enough in Proposition 2.1. $Q_1 Q_2^*$ is a pseudo-differential operator as in (5.5) with symbol ρ

$$\rho(x,y,\xi') = q_1(x,\xi)\overline{q_2(y,\xi)}\mathcal{J}(\xi',x,y)$$

with $\xi = \xi(\xi')$ and $\mathcal{J}$ is the Jacobian of the change of variables.) Therefore, for any $k=0,1,2,\ldots$ there is a constant $C_{k,\phi}$ so that

$$|\rho|_k < C_{k,\phi}|q_1|_k|q_2|_k$$

The Calderón-Vaillancourt theorem applies to show that $Q_1 Q_2^*$ is bounded. In particular the Q_j, $j=1,2$ are bounded operators on $L^2(\mathbf{R}^n)$ and

$$\|Q_j\| \le C|q_j|_k \tag{5.8}$$

(for take $Q_1 = Q_2$).

Example. Recall the operator

$$Ju(x) = \int e^{i\phi(x,\xi)} a(x,\xi)\hat{u}(\xi)\, d_1\xi \tag{5.9}$$

where ϕ and a were defined in (4.6) and (4.5). The preceding discussion shows that JJ^* and hence $JJ^* - 1$ is a pseudo-differential operator with symbol $\rho(x,y,\xi') = a(x,\xi)\overline{a(y,\xi)}\mathcal{J}(\xi',x,y) - 1$, in the notation of the preceding paragraph, and the Calderón-Vaillancourt theorem provides an estimate of the operator norm. Given $r > 0$ it is possible to suppose that

$$\|JJ^* - 1\| \le r \tag{5.10}$$

provided the parameters δ and k in (2.6) and R in (4.2) are suitably chosen. The parameter r will be later be fixed very small during the construction of an analytic extension of the resolvent of H below. Specifying $r < 1$ assures that JJ^* is invertible and J has a one-sided inverse, $J^*(JJ^*)^{-1}$ (which may be a two-sided inverse under appropriate assumptions.)

It is the operators $QT_0(\lambda)^*$ that are of interest in this Section. Recall $T_0(\lambda)^*$ maps $L^2(\mathbf{R}^{n-1})$ to $H^{-s}\otimes L^2$ and Q is initially defined on $\mathcal{S}(\mathbf{R}^n)$ so that it is necessary to check where and how $QT_0(\lambda)^*$ is defined. One convenient way to represent $QT_0(\lambda)^*$ is to first choose $\tilde{\chi} \in \mathcal{S}(\mathbf{R})$ with $\tilde{\chi}(0) = 1$. Then for any $u \in \mathcal{S}(\mathbf{R}^{n-1})$:

$$QT_0(\lambda)^*u(x) = \lim_{\delta\to 0} \frac{1}{\sqrt{2\pi F^n}} \int e^{i\phi(x,\xi)-i(\lambda\xi_1-G(\xi))/F}\tilde{\chi}(\delta\xi_1)q(x,\xi)\hat{u}(\xi_\perp/F)\,d\xi \tag{5.11}$$

for any $u \in \mathcal{S}(\mathbf{R}^{n-1})$ provided the limit exists in $L^2(\mathbf{R}^n)$. The next result gives a simple condition on q that assures the limit exists and defines a bounded operator.

Lemma 1 *Assume $\epsilon_V > 1/2$ in the Hypothesis. Then for any $k \in \mathbf{N}$ there are constants $C > 0$ and $m \in \mathbf{N}$ so that for any operator Q as in (5.7) with symbol q,*

$$\|[\tilde{\chi}_{(-\infty,0)} + \langle x_1\rangle^{-k}\tilde{\chi}_{(-\infty,0)}]Q[|x_1|^k\tilde{\chi}_{(-\infty,0)}]\| \leq C|q|_m. \tag{5.12}$$

Here $\tilde{\chi}_{(0,\infty)}$ and $\tilde{\chi}_{(-\infty,0)}$ are as in (1.3). In addition, for any real λ, the operator

$$(\tilde{\chi}_{(0,\infty)} + \langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,0)})T_0(\lambda)^* \tag{5.13}$$

is bounded as a mapping from $L^2(\mathbf{R}^{n-1})$ to $L^2(\mathbf{R}^n)$ for every $\lambda \in \mathbf{R}$. In particular $[\tilde{\chi}_{(-\infty,0)} + \langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,0)}]QT_0(\lambda)^*$ is a bounded operator from $L^2(\mathbf{R}^{n-1})$ to $L^2(\mathbf{R}^n)$ whenever $|q|_m < \infty$.

Proof: Consider first $T_0(\lambda)$ and (5.13). It will be shown that

$$(\tilde{\chi}_{(1,\infty)} + \langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,1)})T_0(\lambda)^* \tag{5.14}$$

is bounded where $\tilde{\chi}_{(1,\infty)}$ in $C^\infty(\mathbf{R})$ is chosen so that $\tilde{\chi}_{(1,\infty)}(x_1) = 1$ if $x_1 > 3/2$ and $\tilde{\chi}_{(1,\infty)}(x_1) = 0$ if $x_1 < 1/2$ and $\tilde{\chi}_{(-\infty,1)}) = 1 - \tilde{\chi}_{(1,\infty)}$. This implies (5.13). Begin the proof of (5.14) by considering the L^2 inner product

$(\langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,1)}T_0(\lambda)^*u, v)$ for $u \in \mathcal{S}(\mathbf{R}^{n-1})$ and $v \in \mathcal{S}(\mathbf{R}^n)$. By (5.11)

$$\begin{aligned}
&(2\pi F^n)^{1/2}(\langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,1)}(x_1)T_0(\lambda)^*u, v) \\
&= \lim_{\delta\to 0}\iint e^{ix\cdot\xi - i(\lambda\xi_1 - G(\xi))/F}\tilde{\chi}(\delta\xi_1)\langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,1)}(x_1)\overline{v(x)}\hat{u}(\xi_\perp/F)\,d\xi\,dx
\end{aligned}$$

for any $\tilde{\chi} \in \mathcal{S}(\mathbf{R}^n)$ such that $\tilde{\chi}(0) = 1$. Substitute the identity

$$\frac{1}{1+|\xi|^2/F}\left(1 - i\frac{\partial}{\partial\xi_1}\right)e^{iG(\xi)/F} = e^{iG(\xi)/F} \tag{5.15}$$

and integrate by parts in ξ_1:

$$(2\pi F^n)^{1/2}(\langle x_1\rangle^{-1}\chi_{(-\infty,1)}T_0(\lambda)^*u, v) = (\Psi w, v)$$

where w is given by its Fourier transform,

$$\hat{w}(\xi) = \langle\xi_1\rangle^{-2}\exp(iG(\xi)/F)\hat{u}(\xi_\perp),$$

(which is in $L^2(\mathbf{R}^n)$) and Ψ is a pseudo-differential operator (5.5) which is bounded by the Calderón-Vaillancourt theorem. And so $\langle x_1\rangle^{-1}\chi_{(-\infty,1)}T_0(\lambda)^*$ is bounded on $L^2(\mathbf{R}^{n-1})$ to $L^2(\mathbf{R}^n)$. The argument for $\chi_{(1,\infty)}T_0(\lambda)^*$ is very similar with the minor difference that the integration by parts involves integrating $\exp(ix_1\xi_1 + iG(\xi)/F)$ this time (instead of (5.15)).

To establish (5.12), it will be shown that

$$\|[\tilde{\chi}_{(-\infty,0)} + \langle x_1\rangle^{-k}\tilde{\chi}_{(-\infty,0)}]Q(|x_1|^k\tilde{\chi}_{(-\infty,-1)})\| \le C|q_1|_m. \tag{5.16}$$

where $\tilde{\chi}_{(-\infty,-1)})(x_1) = \tilde{\chi}_{(1,\infty)})(-x_1)$; clearly (5.12) and (5.16) are equivalent. Integrate by parts k times in ξ_1

$$\begin{aligned}
&Qx_1^k\tilde{\chi}_{(-\infty,-1)}(x_1)u(x) \\
&= (-i)^k\tilde{\chi}_{(-\infty,0)}(x_1)\int\frac{\partial^k}{\partial\xi_1^k}\left(e^{i\phi(x,\xi)}q(x,\xi)\right)(\tilde{\chi}_{(-\infty,-1)}u)\hat{\ }(\xi)\,d\xi \\
&\quad + i\tilde{\chi}_{(0,\infty)}(x_1)\text{Os-}\iint e^{-iy\cdot\xi}\frac{\partial^k}{\partial\xi_1^k}\left(e^{i\theta(x,\xi)}q(x,\xi)\right) \\
&\qquad\qquad \left[\frac{y_1+x_1}{y_1}\right]^k\tilde{\chi}_{(-\infty,-1)}(y_1+x_1)u(y+x)\,dyd\xi
\end{aligned}$$

where $\theta(x,\xi) = \phi(x,\xi) - x \cdot \xi$. Multiply the above equation by $[\tilde{\chi}_{(-\infty,0)} + \langle x_1 \rangle^{-k} \tilde{\chi}_{(-\infty,0)}]$: then (5.16) follows from the Calderón-Vaillancourt theorem and (5.8). (Here and only here $e^{i\theta}$ is regarded as part of the symbol and not the phase; it is in $C_b^\infty(\mathbf{R}^n)$ by Proposition 2.1.) □

The main result of this Section, Proposition 2 is comparable to Yajima's [34, Lemma 2.1]. It assures an analytic continuation of $QT_0(\lambda)^*$ as a function of λ under appropriate assumptions on Q. It is convenient therefore to introduce a class of symbols q for these operators. Briefly $\mathcal{B}$ will be defined so that it includes the symbol a and a_e/h_μ (h_μ was defined in (1.2)); whereas $\mathcal{B}_\mathcal{A} \subset \mathcal{B}$ includes $a_\mathcal{A}$. More precisely, for each $R, \delta > 0$, let

$$\begin{aligned} \Omega_{R,\delta} \;\; = \;\; & \mathbf{R}^{2n} \cup \{(x,\xi) \in \mathbf{C}^{2n} : |\Im x_1| \le \delta \langle \Re x_1 + K_\sigma \rangle, \Re x_1 < -R, \\ & |\Im x_\perp|^2 + |\Im \xi_\perp|^2 < \delta^2, |\Im \xi_1| < \delta\} \end{aligned}$$

($\Omega_{R,\delta}$ is small enough to be contained in the domains of ϕ, a, a_e $a_\mathcal{A}$ but large enough to enable the change of path of integration arguments of the next two Propositions.) Let $C_b^M(\Omega_{R,\delta})$ denote the space of all functions continuous on $\Omega_{R,\delta}$ and bounded along with their first M partial derivatives in the real and imaginary parts of the $2n$ variables. For each integer $M > 0$ and $\delta > 0$ define $\mathcal{B}(M,\delta)$ to be the set of all functions q in $C^\infty(\Omega_{R,\delta}) \cap C_b^M(\Omega_{R,\delta})$ for some $R = R(q) > 0$ such that for each fixed x_1, $q(x_1, x_\perp, \xi)$ is analytic in $(x_\perp, \xi)$ on the cross section of $\Omega_{R,\delta} \cap \{|\Re \xi_1| > \kappa\}$ for some positive $\kappa = \kappa(q)$. (K_σ is from Proposition 3.1 and is fixed.)

Define $\mathcal{B}_\mathcal{A}(M,\delta) \subset \mathcal{B}(M,\delta)$ to include those q which are analytic in all $2n$ variables on

$$\Omega_{R,\delta} \cap \{|\Re \xi_1| > \kappa\} \cap \{\Re x_1 < -R\}$$

and, for fixed ξ_1, $|\xi_1| \le \kappa$ are analytic in the $2n-1$ variables $(x, \xi_\perp)$ on the cross section of $\Omega_{R,\delta} \cap \{\Re x_1 < -R\}$ for some $R = R(q) > 0$ and $\kappa = \kappa(q) > 0$. Define further, for any integer $m > 0$,

$$\|q\|_m = \sup\{|\partial^\alpha \overline{\partial}^\beta q(x,\xi)| : (x,\xi) \in \Omega_{R,\delta}, |\alpha + \beta| \le m, \alpha, \beta \in \mathbf{N}_0^{2n}\}. \quad (5.17)$$

where ∂^α and $\overline{\partial}^\beta$ are the higher order derivatives in the $2n$ complex variables (x,ξ) corresponding to those introduced in (3.2). (Reference to R and δ is omitted from the notation $\|\cdot\|_m$.) Finally let $\mathcal{B} = \cup_{M,\delta>0} \mathcal{B}(M,\delta)$ and $\mathcal{B}_\mathcal{A} = \cup_{M,\delta>0} \mathcal{B}_\mathcal{A}(M,\delta)$. Then, provided $\mu > 0$ is small enough

$$a, a_e/h_\mu \in \mathcal{B} \qquad a_\mathcal{A} \in \mathcal{B}_\mathcal{A}.$$

Observe that, for any $M > 0$, it is possible to choose $\mu > 0$ so small so that a_e/h_μ is in $\mathcal{B}(M, \delta)$ for some $\delta > 0$. Also $\mathcal{B}$ and $\mathcal{B}_\mathcal{A}$ are algebras.

Proposition 2 *Assume $\epsilon_V > 1/2$ in the Hypothesis.*

(a). Suppose $\mu > 0$ and $k \in \mathbf{N}_0$. Then there is $M = M(k)$ so that, for any operator Q as in (5.7) with symbol $q \in \mathcal{B}(M, \delta)$, for some $\delta > 0$

$$(-\Delta)^k h_\mu Q T_0(\lambda)^* \text{ and } H_0^k h_\mu Q T_0(\lambda)^* \tag{5.18}$$

extend to entire functions of λ taking values in the space of bounded operators from $L^2(\mathbf{R}^{n-1})$ to $L^2(\mathbf{R}^n)$. Denote the extensions by the same symbols. For any compact $\mathcal{K} \subseteq \mathbf{C}$, there is a constant C not depending on q so that, for all $\lambda \in \mathcal{K}$,

$$\|(-\Delta)^k h_\mu Q T_0(\lambda)^*\| + \|H_0^k h_\mu Q T_0(\lambda)^*\| < C\|q\|_M \tag{5.19}$$

(b). For any $\nu > 0$, and Q with symbol $q \in \mathcal{B}$, $Qe^{-\nu\langle D_1\rangle} T_0(\lambda)^*$ extends to $\{|\Im\lambda| < \nu\}$ as an analytic operator valued function bounded from $L^2(\mathbf{R}^{n-1})$ to $L^2(\mathbf{R}^n)$. (The extension is denoted by the same symbol.) Moreover if $0 < \nu' < \nu$ then there exists $C > 0$ and $m \in \mathbf{N}$ so that for all λ, $|\Im\lambda| \le \nu'$

$$\|Qe^{-\nu\langle D_1\rangle} T_0(\lambda)^*\| < C\|q\|_m$$

Proof of Proposition 2: Establish Part (b) first. Clearly $Qe^{-\nu\langle D_1\rangle}$ is bounded from $H^{-s} \otimes L^2$ to $L^2(\mathbf{R}^n)$ for any s and the representation (5.11) is valid. (The limit in (5.11) can be taken under the integral sign.) From (5.11), it is clear that $Qe^{-\nu\langle D_1\rangle} T_0(\lambda)^*$ extends analytically to $\{\lambda \in \mathbf{C} : |\Im\lambda| \le \nu' < \nu\}$ and is bounded as claimed; (see (5.8)).

For the proof of Part (a) recall that $h_\mu Q T_0(\lambda)$ is bounded by Lemma 1 and the expansion (5.11) is valid. Apply Cauchy's Theorem in (5.11) of $h_\mu Q T_0(\lambda)^* u\,(x)$ to change the path of integration for ξ_1 from the real line to a path which is the graph $\xi_1 + i\zeta(\xi_1)$ of a nonnegative C^∞ function $\zeta(\xi_1)$ which is 0 if $|\xi_1| < \kappa$ and is the constant μ if $|\xi_1| > 2\kappa$ where $\kappa > 0$ is chosen so that ϕ and q are analytic for $|\Re\xi_1| > \kappa$. Therefore, letting $\theta(x,\xi) = \phi(x,\xi) - x\cdot\xi$,

$$\begin{aligned} &(2\pi)^{1/2} F^{n/2} h_\mu Q T_0(\lambda)^* v\,(x) \\ &= \int_{\mathbf{R}^{n-1}} \int e^{ix\cdot\xi + i\theta(x,\xi+i\zeta\mathbf{e}_1)} e^{-i\lambda\xi_1 + iG(\xi+i\zeta\mathbf{e}_1)/F} e^{\lambda\zeta - x_1\zeta} h_\mu(x_1) \\ &\qquad q(x, \xi + i\zeta\mathbf{e}_1) d(\xi_1 + i\zeta) \hat{v}(\xi_\perp/F)\, d\xi_\perp. \end{aligned} \tag{5.20}$$

The exponential factor $e^{iG/F}$ decays rapidly when $\zeta > 0$ because $iG(\xi + i\zeta\mathbf{e}_1) = iG(\xi) - i\xi_1\zeta^2 + \zeta^3/3 - \zeta|\xi|^2$ so that the above integral exists absolutely. Since $e^{-x_1\zeta}h_\mu(x_1)q(x, \xi + i\zeta\mathbf{e}_1)$ is bounded along with its derivatives up to order M, the decay of e^{iG} assures that $h_\mu QT_0(\lambda)^*$ extends in λ into the complex plane as an entire function taking values in the space of bounded operators of $L^2(\mathbf{R}^{n-1})$ to $L^2(\mathbf{R}^n)$. Check (5.19) in the case $k = 0$ next. It follows from (5.8) provided one checks that $\theta(x, \xi + i\zeta\mathbf{e}_1)$ can be replaced by $\theta(x, \xi)$ in (5.20) . Since $\exp(i(\theta(x, \xi + i\zeta\mathbf{e}_1) - \theta(x, \xi)))$ is in $\mathcal{B}_\mathcal{A}$, it can be incorporated into the symbol.

It remains to check (5.19) when $k > 0$. Consider the case $k = 1$ there and begin with $(-\Delta)h_\mu QT_0(\lambda)^*$. If the symbol q happened to have compact support in ξ then this case would follow from the $k = 0$ case already treated. It is possible therefore to assume that the symbol is supported on a region that does not include $\xi = 0$. Substitute the identity below

$$
\begin{aligned}
& e^{ix\cdot\xi + iG(\xi)/F - \zeta|\xi|^2/F} \\
&= \tilde{\chi}_{(-\infty,0)}(Rx_1)e^{ix\cdot\xi}\frac{F}{|\xi|^2(i-\zeta') - 2\zeta\xi_1}\frac{\partial}{\partial\xi_1}e^{iG(\xi)/F - \zeta|\xi|^2/F} \\
&\quad + \tilde{\chi}_{(0,\infty)}(Rx_1)\frac{F}{iFx_1 + |\xi|^2(i-\zeta') - 2\zeta\xi_1}\frac{\partial}{\partial\xi_1}e^{ix\cdot\xi + iG(\xi)/F - \zeta|\xi|^2/F}
\end{aligned}
$$

into the above expansion for $h_\mu QT_0(\lambda)^*v\,(x)$. Here R is a constant chosen large enough to avoid division by 0. ($\tilde{\chi}_{(-\infty,0)}$ was defined in (1.3).) Integrate by parts in the ξ_1 variable. It then follows that $\Delta h_\mu QT_0(\lambda)^*$ is bounded and the operator bound (5.19) in this case follows just as (5.8) above. To complete the argument for the $k = 1$ case, it suffices to compute the operator norm of $x_1 h_\mu QT_0(\lambda)^*$. In view of the exponential decay of h_μ, it suffices to show that $\chi_{(1,\infty)}(x_1)x_1 Qh_\mu T_0(\lambda)^*$ is bounded. The same integration by parts argument as in the first half of the $k = 1$ case applies again here. This completes the proof of the case $k = 1$; the general case follows by repeated integrations by parts. □

In Proposition 2 above it was shown, for example, that $h_\mu QT_0(\lambda)^*$ and $Qe^{-\nu\langle D_1\rangle}T_0(\lambda)^*$ extend analytically provided $Q \in \mathcal{B}$. In the next Section there will be occasion to consider operators of the form $h_\mu Q_2 Q_1^* QT_0(\lambda)^*$ and $e^{-\nu\langle D_1\rangle}Q_1^* QT_0(\lambda)^*$ where $Q_1, Q_2 \in \mathcal{B}$. Can they be extended analytically? Under somewhat more restrictive assumptions the answer is yes; this fact follows from the preceding Proposition 2 by way of the Proposition below.

Proposition 3 *Assume that Q_1 and Q_2 are two operators of the form (5.7) with symbols $q_1, q_2 \in \mathcal{B}$.*

(a). Then, for any $\nu > 0$, and any sufficiently small $\mu > 0$, there are Q_k in $\mathcal{B}$, $3 \le k \le 10$ so that

$$Q_1 Q_2^* h_\mu = \sum_{j=1,2} h_\mu Q_{4j-1} Q_{4j}^* + Q_{4j+1} e^{-\nu\langle D_1\rangle} Q_{4j+2}^*$$

If in addition q_1 is in $\mathcal{B}_{\mathcal{A}}$ (resp. if q_2 is in $\mathcal{B}_{\mathcal{A}}$) then q_3, q_5, q_7, and q_9 (resp. q_4, q_6, q_8, and q_{10}) are in $\mathcal{B}_{\mathcal{A}}$.

(b). Suppose now that $q_1, q_2 \in \mathcal{B}_{\mathcal{A}}$. Then, for each $\mu, \nu > 0$ there are Q_k in $\mathcal{B}_{\mathcal{A}}$, $11 \le k \le 18$ so that

$$Q_1^* Q_2 e^{-\nu\langle D_1\rangle} = \Sigma_{j=3,4} e^{-\nu\langle D_1\rangle} Q_{4j-1}^* Q_{4j} + Q_{4j+1}^* h_\mu Q_{4j+2}$$

Denote the symbols of Q_k by q_k. Then, for any integer $m > 0$, there is a constant $C > 0$, not depending on q_1 or q_2 so that, for $2 \le j \le 9$,

$$\|q_{2j-1}\|_m < C\|q_1\|_m \quad \text{and} \quad \|q_{2j}\|_m < C\|q_2\|_m.$$

Proof. Establish Part (b) first. Consider $\tilde{\chi}_{(-\infty,0)} \mathcal{F} Q_1^* Q_2 e^{-\nu\langle D_1\rangle} u$ where $\mathcal{F}$ denotes the Fourier transform and where $u \in \mathcal{S}(\mathbf{R}^n)$. By (5.7) and a change in the contour of integration in the x_1 variable one has

$$\begin{aligned}
&\tilde{\chi}_{(-\infty,0)}(\xi_1) \mathcal{F} Q_1^* Q_2 e^{-\nu\langle D_1\rangle} u(\xi) \\
&= \tilde{\chi}_{(-\infty,0)}(\xi_1) \int e^{-i\phi(x+i\omega(x_1)\mathbf{e}_1,\xi)} \overline{q_1(x - i\omega(x_1)\mathbf{e}_1, \xi)} \\
&\qquad \int e^{i\phi(x+i\omega(x_1)\mathbf{e}_1,\zeta)} q_2(x + i\omega(x_1)\mathbf{e}_1, \zeta) e^{-\nu\langle\zeta_1\rangle} \hat{u}(\zeta)\, d\zeta dx. \qquad (5.21)
\end{aligned}$$

The new path of integration is the graph $x_1 + i\omega(x_1)$ of a C^∞ positive function ω where $\omega(x_1) = 0$ if $x_1 > -R$ and is $\omega(x_1) = \nu$ if $x_1 < -R'$ where $R' > R > 0$ are chosen so large that the portion of the graph of $\omega(x_1)$ for which $x_1 < -R$ lies within the domains of analyticity of q_1, q_2 and ϕ.

Introduce the decomposition $1 = \tilde{\chi}_{(-\infty,-R')}(x_1) + \tilde{\chi}_{(-R',\infty)}(x_1)$ into the integral over x in (5.21) where $\tilde{\chi}_{(-\infty,-R')}$ (resp. $\tilde{\chi}_{(-R',\infty)}$) is a C^∞ function supported on $(-\infty, -R')$ (resp $(-R'-1, \infty)$). On the support of $\tilde{\chi}_{(-\infty,-R')}$ (where $\omega(x_1) = \nu$), the change in the phase due to the change of path is

$$\begin{aligned}&\phi(x + i\omega(x_1)\mathbf{e}_1, \zeta) - \phi(x + i\omega(x_1)\mathbf{e}_1, \xi) - \phi(x, \zeta) + \phi(x, \xi) \\ &\quad = \; -i\nu(\xi_1 - \zeta_1) + \theta(x + i\nu\mathbf{e}_1, \zeta) - \theta(x, \zeta) - \theta(x + i\nu\mathbf{e}_1, \xi) + \theta(x, \xi)\end{aligned}$$

where $\theta(x, \xi) = \phi(x, \xi) - x \cdot \xi$. Since $\exp(i(\theta(x, \xi) - \theta(x + i\nu\mathbf{e}_1, \xi)))$ is in $\mathcal{B}_{\mathcal{A}}$, it can be regarded as part of the symbol. Similarly

$$e^{\nu\langle\xi_1\rangle}\tilde{\chi}_{(-\infty,0)}(\xi_1)e^{\nu\xi_1} \quad \text{and} \quad e^{-\nu\zeta_1}e^{-\nu\langle\zeta_1\rangle}$$

both belong to $\mathcal{B}_{\mathcal{A}}$ and they too may be incorporated into the symbols so that

$$\tilde{\chi}_{(-\infty,0)}(D_1)Q_1^*\tilde{\chi}_{(-\infty,-R')}Q_2e^{-\nu\langle D_1\rangle}$$

is of the form $e^{-\nu\langle D_1\rangle}Q_3^*Q_4$ of the statement of this result. The other portion of (5.21) where $\tilde{\chi}_{(-R',\infty)}$ replaces $\tilde{\chi}_{(-\infty,-R')}$ is clearly of the form $Q_5^*h_\mu Q_6$. To complete the proof of Part (b) it remains to show $\tilde{\chi}_{(0,\infty)}\mathcal{F}Q_1^*Q_2e^{-\nu\langle D_1\rangle}u$ also has the required expansion. The argument is exactly parallel to that above except that the path of integration becomes $x_1 - i\omega(x_1)$ this time. This establishes Part (b) and the estimates on the symbols $\|q_k\|_m$, $3 \le k \le 10$ are routinely verified.

The proof of Part (a) is similar. Consider $\tilde{\chi}_{(-\infty,0)}(x_1)Q_1Q_2^*h_\mu u(x)$. Again Cauchy's theorem applies to change the path of integration, this time in the ξ_1 variable, from along the real axis to along the graph $\xi_1 + i\zeta_1(\xi_1)$ of a C^∞ function ζ which is 0 if $|\xi_1| < 3\kappa/2$ and is $-\mu$ when $|\xi_1| > 5\kappa/2$. (Therefore μ must be small enough to for the path to be in the domain of analyticity of q_1, q_2 and ϕ.) Introduce $\tilde{\chi}(\xi_1)$ in $C_0^\infty(\mathbf{R})$ which is one if $|\xi_1| < 5\kappa/2$. Then

$$\begin{aligned}&\tilde{\chi}_{(-\infty,0)}(x_1)Q_1(1 - \tilde{\chi}(D_1))Q_2^*h_\mu u(x) \\ &\quad = \; \tilde{\chi}_{(-\infty,0)}(x_1)\int e^{i\phi(x,\xi-i\mu\mathbf{e}_1)}q_1(x, \xi - i\mu\mathbf{e}_1)) \\ &\qquad \int e^{-i\phi(y,\xi-i\mu\mathbf{e}_1)}\overline{q_2(y, \xi + i\mu\mathbf{e}_1)}(1 - \tilde{\chi}(\xi_1))h_\mu(y_1)u(y)dy\,d\xi\end{aligned}$$

The change of phase due to the change of path is

$$\begin{aligned}&\phi(x, \xi - i\mu\mathbf{e}_1) - \phi(y, \xi - i\mu\mathbf{e}_1) - \phi(x, \xi) + \phi(y, \xi) \\ &\quad = \; -i(x_1 - y_1)\mu + \theta(x, \xi - i\mu\mathbf{e}_1) - \theta(x, \xi) - (\theta(y, \xi - i\mu\mathbf{e}_1) - \theta(y, \xi)).\end{aligned}$$

The factor $\exp\theta(x,\xi - i\mu\mathbf{e}_1) - \theta(x,\xi)$ is in $\mathcal{B}_{\mathcal{A}}$ whereas $e^{-\mu y_1}h_\mu(y_1)$ is in $C_b^\infty(\Omega_{R,\delta})$ for suitable $R,\delta > 0$ and so both factors may be incorporated into the symbol. Therefore $\tilde{\chi}_{(-\infty,0)}Q_1(1-\tilde{\chi}(D_1))Q_2^* h_\mu$ is of the form $h_\mu Q_3 Q_4^*$. Obviously $\tilde{\chi}_{(0,\infty)}Q_1(1-\tilde{\chi}(D_1))Q_2^* h_\mu$ is of the same form. As for $Q_1\tilde{\chi}(D_1)Q_2^*$, it is of the form $Q_5 e^{-\nu\langle D_1\rangle}Q_6$ for any $\nu > 0$ which verifies Part (a). □

6 Continuation of the Resolvents.

It is possible to analytically continue the resolvent $R_0(z) = (H_0 - z)^{-1}$ across the real axis whereas the resolvent $R(z) = (H - z)^{-1}$ has a meromorphic continuation under certain assumptions which will be made precise in this Section. The result below, for R_0, is an adaptation of Yajima's [34, Corollary 2.3]

Proposition 1 *Assume $\epsilon_V > 1/2$. Suppose that Q_1 and Q_2 are integral operators as in (5.7) with symbols q_1 and q_2 in $\mathcal{B}$ and $\mu,\nu > 0$. Distinguish four possible choices for the definitions of P_1 and P_2:*

1. $P_1 = h_\mu Q_1; P_2 = h_\mu Q_2,$ 2. $P_1 = h_\mu Q_1; P_2 = Q_2 e^{-\nu\langle D_1\rangle},$
3. $P_1 = Q_1 e^{-\nu\langle D_1\rangle}; P_2 = h_\mu Q_2,$ *or* 4. $P_1 = Q_1 e^{-\nu\langle D_1\rangle}; P_2 = Q_2 e^{-\nu\langle D_1\rangle}.$

Then, in any case, $P_1 R_0(z) P_2^*$ has an analytic extension from $\mathbf{C}_\pm = \{\pm\Im z > 0\}$ to $\{\pm\Im z > -\nu\}$ (and in fact to all of $\mathbf{C}$ in case 1) as an operator on $L^2(\mathbf{R}^n)$. The extensions are denoted $P_1 R_{0,\pm}(z) P_2^*$. Additionally

$$P_1 R_{0,\pm}(z) P_2^* = P_1 R_0(z) P_2^* \pm 2\pi i P_1 T_0(z)^* T_0(z) P_2^*$$

for $\pm\Im z < 0$. For any compact subset $\mathcal{K}$ of $\{z : \pm\Im z > -\nu\}$ (or $\mathcal{K} \subseteq \mathbf{C}$ in case 1), there is a constant C and $m = 0,1,2,\ldots$ not depending on q_1 or q_2 so that,

$$\sup_{z\in\mathcal{K}} \|P_1 R_{0,\pm}(z) P_2^*\| < C\|q_1\|_m \|q_2\|_m.$$

In cases 1 and 2, if one further supposes that Q_1 is bounded as a mapping from the domain $D(H_0)$ of H_0 (with the graph norm) to itself then $h_\mu Q_1 R_{0,\pm}(z) P_2^*$, is bounded from $L^2(\mathbf{R}^n)$ to $D(H_0)$ and, for $\mathcal{K}$, C and m as above

$$\sup_{z\in\mathcal{K}} \|H_0 h_\mu Q_1 R_{0,\pm}(z) P_2^*\| < C\|q_1\|_m \|q_2\|_m$$

Remark. Observe that $T_0(z)P_2^* = (P_2T_0(\overline{z})^*)^*$ has an analytic extension by Proposition 5.2.

Proof of Proposition 1. It suffices to consider the "+" case ($\Im z > 0$) because the other case is similar. Let I denote a bounded open interval and I^c its complement. Then the operator

$$P_1R_0(z)E_0(I^c)P_2^* = [P_1R_0(i)][(H_0 - i)R_0(z)E_0(I^c)]P_2^*$$

can be continued across the interval I because $(H_0-i)R_0(z)E_0(I^c)$ can be as an operator on $L^2(\mathbf{R}^n)$ and $P_1R_0(i)$ is bounded on $L^2(\mathbf{R}^n)$. Moreover if Q_1 is bounded on $D(H_0)$ then the continuation of $P_1R_0(z)E_0(I^c)P_2^*$ is bounded from $L^2(\mathbf{R}^n)$ to $D(H_0)$ provided z stays a fixed distance from I^c.

On the other hand, by (5.4)

$$(P_1R_0(z)E_0(I)P_2^*u, v) = \int_I \frac{(T_0(\lambda)P_2^*u, T_0(\lambda)P_1^*v)}{\lambda - z}\, d\lambda$$

for $u, v \in \mathcal{S}(\mathbf{R}^n)$ and $z \in \mathbf{C}_+$. (Q_1^* and Q_2^* map $\mathcal{S}(\mathbf{R}^n)$ to itself.) According to Proposition 5.2, $P_jT_0(\lambda)^*$, $j = 1, 2$ both extend analytically below the real axis so that the path of integration above is arbitrary between the endpoints of I. Consequently the above expression allows one to extend $P_1R_0(z)E_0(I)P_2^*$ to below the real axis and the extension is

$$P_1R_0(z)E_0(I)P_2^* + 2\pi i P_1T_0(z)^*T_0(z)P_2^*.$$

The final statement of the Proposition follows because $h_\mu Q_1T_0(z)^*$ is bounded as a mapping from $L^2(\mathbf{R}^n)$ to $D(H_0)$ by Proposition 5.2. This completes the proof. □

Next is the analogous result for H. This time the extension of $R(z)$ is *meromorphic* which means that in any compact subset of the complex plane there are finitely many poles and at any pole z_0, $R(z)$ can be expanded in a Laurent series in powers of $z - z_0$ where the coefficients of the negative powers are finite rank operators and only finitely many are nonzero. There is a result sometimes known as the "analytic Fredholm" theorem about meromorphic operators which can be stated as follows: *If $P(z)$ is an analytic operator valued function defined for z in a connected domain $\Omega \subseteq \mathbf{C}$ such that $P(z)$ is compact for all z and $(1 - P(z))^{-1}$ exists for some $z \in \Omega$ then $(1 - P(z))^{-1}$ is meromorphic on Ω.* For a proof see [20, Theorem VII.1.9] or [29, Theorem VI.14].

Since the continuation result for $R(z)$ involves a few technical arguments, some motivation is appropriate. Recall, from the Example §5, that JJ^* is invertible. Expand the resolvent as

$$\begin{aligned} R(z) &= R(z)JJ^*(JJ^*)^{-1} \\ &= \{JR_0(z) - R(z)[HJ - JH_0]R_0(z)\}J^*(JJ^*)^{-1}. \end{aligned}$$

Introducing the notation $V_J = HJ - JH_0$ and $K = (JJ^*)^{-1}J$ one has

$$R(z)(1 + V_J R_0(z)K^*) = JR_0(z)K^*. \tag{6.1}$$

For the purpose of motivation, suppose for the moment that $V_J = h_\mu^2 Q$ for some Q in $\mathcal{B}$ and $0 < \mu$. Multiplying on the left and right by h_μ, in equation (6.1), one has

$$h_\mu R(z)h_\mu(1 + h_\mu Q R_0(z)K^* h_\mu) = h_\mu J R_0(z) K^* h_\mu.$$

Suppose not only that Q is in $\mathcal{B}$ but K is also and that Q is H_0-compact. Then $1+h_\mu Q R_0(z)K^*h_\mu$ has a meromorphic inverse by the analytic Fredholm theorem stated above, so that $h_\mu R(z)h_\mu$ has a meromorphic continuation. The most serious error in the argument is that $V_J \neq h_\mu^2 Q$ because of the cutoffs introduced in the definition of ϕ and a. However V_J is close in operator norm to an operator of the form $h_\mu^2 Q$. Also K is not demonstrably in $\mathcal{B}$ but again it is close in operator norm.

So what is V_J? It is $V_J = V_e J + H_{\mathcal{A}} J - JH_0$ and recall $H_{\mathcal{A}}J - JH_0$ is an operator of the form (5.7) with symbol t in (2.1). The symbol t can be written as $t = h_{\mu_0} t_1 + t_2$ where t_1, t_2 are in $\mathcal{B}$ and t_2 is compactly supported in ξ_1 and $\mu_0 > 0$ is small enough. In fact for any $M > 0$, there is $\mu_0, \delta > 0$ so that $t_1 \in \mathcal{B}(M, \delta)$. It will be convenient to express t_2 in terms of ϕ^+ of Proposition 2.1 and b of Proposition 3.1 but the explicit expression for t_1 is omitted for brevity. The expression for t_2 is written in terms of an auxiliary function t_2^+: $t_2(x,\xi) = t_2^+(x,\xi) + \overline{t_2^+(\overline{x}, -\overline{\xi})}$ where

$$\begin{aligned} t_2^+(x,\xi) = &-(F/\kappa_\sigma)\theta^+(x,\xi)\tilde{\chi}'_{(-\infty,0)}(\Re\xi_1/\kappa_\sigma) \\ &+ i(F/\kappa_\sigma)(b_{\mathcal{A}}(x,\xi) - 1)\tilde{\chi}'_{(-\infty,-2)}(\Re\xi_1/\kappa_\sigma) \\ &+ |\nabla_x\theta^+(x,\xi)|^2[\tilde{\chi}^2_{(-\infty,0)}(\Re\xi_1/\kappa_\sigma) - \tilde{\chi}_{(-\infty,0)}(\Re\xi_1/\kappa_\sigma)] \\ &+ p(x,\xi)\tilde{\chi}_{(-\infty,0)}(\Re\xi_1/\kappa_\sigma)(1 - \tilde{\chi}_{(-\infty,-2)}(\Re\xi_1/\kappa_\sigma)) \\ &+ \nabla_x\theta^+(x,\xi)\cdot\nabla_x\theta^-(x,\xi)\tilde{\chi}_{(-\infty,0)}(\Re\xi_1/\kappa_\sigma)\tilde{\chi}_{(0,\infty)}(\Re\xi_1/\kappa_\sigma) \end{aligned}$$

where p is given by (2.2) and $\theta^+(x,\xi) = \phi^+(x,\xi) - x\cdot\xi$ and $\theta^-(x,\xi) = -\theta^+(x,-\xi)$. Therefore t_2 is in $\mathcal{B}_{\mathcal{A}}$. Denote the operators with symbols t_1 and t_2 (defined by (5.7)) by V_1 and V_2 so that $H_{\mathcal{A}}J - JH_0 = h_{\mu_0}V_1 + V_2$. Finally, it is convenient to cutoff t_2 to be 0 if $x_1 > -R$ where $R > 0$ is suitably large. Explicitly, replace t_2 by $\tilde{\chi}_{(-\infty,-2)}(x_1/R)t_2(x,\xi)$ where R is chosen so large that

$$\|t_2\|_m < \tilde{r} \tag{6.2}$$

where $\tilde{r} > 0$, $m = 0, 1, 2, \ldots$ are to be specified below. It is here that the hypothesis $\epsilon_V > 1/2$ is needed, specifically for the first term of t_2^+ involving θ^+: see Proposition 2.1 and (5.8). The remainder of t_2 can be incorporated into $h_\mu(x_1)t_1$. For simplicity the notations t_1 and t_2 will not be changed.

The motivational argument, two paragraphs above, can be buttressed as follows. Provided $|\Im z|$ is large enough $1 + V_2R_0(z)K^*$ is invertible so that, from (6.1)

$$\begin{aligned} & R(z)[1 + h_{\mu_0}(V_1 + V_0J)R_0(z)K^*(1 + V_2R_0(z)K^*)^{-1}] \\ = \;& JR_0(z)K^*(1 + V_2R_0(z)K^*)^{-1} \end{aligned} \tag{6.3}$$

For notational convenience it has been supposed that $\mu_V = \mu_0$: re-choose μ_0 and V_0 if necessary. Now it should be possible to multiply on both sides by h_μ, and solve for $h_\mu R(z)h_\mu$. This idea leads to:

Theorem 2 *Suppose $\epsilon_V > 1/2$ in the Hypotheses. Then, for any $\mu > 0$, each of the operators*

$$h_\mu R(z)h_\mu, \quad V_2^*R(z)h_\mu, \quad h_\mu R(z)V_2 \quad \text{and} \quad V_2^*R(z)V_2$$

has a meromorphic extension, from $\mathbf{C}_\pm$ to $\mathbf{C}$ as a bounded operator on $L^2(\mathbf{R}^n)$.

Proof. It suffices to consider the "+" ($\Im z > 0$) case; the other case is similar. The first concern is that K is not an operator with symbol in $\mathcal{B}$ so that Proposition 5.2 is not immediately applicable in the expansion (6.3) to conclude a meromorphic continuation. As a remedy two expansions (6.4), (6.5) below will be derived that essentially approximate K by operators to which Proposition 5.2 applies. Expanding $K^* = J^*(JJ^*)^{-1}$ in a Neumann series, one has

$$K^* = \sum_{l=0}^{\infty} J^*(1 - JJ^*)^l$$

which is convergent by (5.10). It will be shown that, for any integer $l > 0$ and $\nu > 0$ there exist bounded operators Q_k in $\mathcal{B}$ and A_k and $\mu > 0$ so that

$$J^*(1 - JJ^*)^l V_2 = \sum_{j \geq 1} Q_{4j}^* h_\mu A_{2j} Q_{4j+1} + e^{-\nu \langle D_1 \rangle} Q_{4j+2}^* A_{2j+1} Q_{4j+3} \qquad (6.4)$$

where $\|A_k\| < C_\nu^{l+1} r^l$, where r is the constant of (5.10) and C_ν does not depend on k or l. Moreover, the symbols q_k of Q_k satisfy the following bounds: for each m, $\|q_{4j+1}\|_m + \|q_{4j+3}\|_m < C_m \tilde{r}$ and $\|q_{4j}\|_m + \|q_{4j+2}\| < C_m$ where $\tilde{r}$ is the constant of (6.2) and again $C_m > 0$ does not depend on l or j. Finally the number of terms in (6.4) is $O(8^l)$. (The Q_k and A_k depend on l, of course, although this is not indicated in the notation.)

To verify (6.4) recall that the symbol a of J decomposes as $a = a_{\mathcal{A}} + a_e$ (Corollary 4.2) so that $J = J_{\mathcal{A}} + J_e$ where $J_{\mathcal{A}}$ (resp. J_e) has symbol $a_{\mathcal{A}}$ (resp. a_e). For any $R > 0$

$$J^* = J^*[1 - \tilde{\chi}_{(-\infty,-2)}(x_1/R)] + J_e^* \tilde{\chi}_{(-\infty,-2)}(x_1/R) + J_{\mathcal{A}}^* \tilde{\chi}_{(-\infty,-2)}(x_1/R)$$

If one now multiplies on the right side by $(1 - JJ^*)^l V_2$ then the first two terms on the right side are of the form of the operators $Q_{4j}^* h_\mu A_{2j} Q_{4j+1}$ on the right side of (6.4). (Recall from §5 that a_e/h_μ is in $\mathcal{B}$ provided $\mu > 0$ is small enough.) Consider therefore

$$\begin{aligned}
&J_{\mathcal{A}}^* \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1 - JJ^*)^l V_2 \\
&= \; J_{\mathcal{A}}^* \tilde{\chi}_{(-\infty,-2)}(x_1/R) J_e J^* (1 - JJ^*)^{l-1} V_2 \\
&+ \; J_{\mathcal{A}}^* \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1 - J_{\mathcal{A}} J^*)[1 - \tilde{\chi}_{(-\infty,-2)}(x_1/R)](1 - JJ^*)^{l-1} V_2 \\
&+ \; J_{\mathcal{A}}^* \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1 - J_{\mathcal{A}} J_e^*) \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1 - JJ^*)^{l-1} V_2 \\
&+ \; J_{\mathcal{A}}^* \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1 - J_{\mathcal{A}} J_{\mathcal{A}}^*) \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1 - JJ^*)^{l-1} V_2.
\end{aligned}$$

The first term on the right side of the above equation is already of the form $Q_{4j}^* h_\mu A_{2j} Q_{4j+1}$. Moreover both $\|\tilde{\chi}_{(-\infty,-2)}(x_1/R) J_e J^*\| < r/2$ and also $\|\tilde{\chi}_{(-\infty,-2)}(x_1/R) J_e J_{\mathcal{A}}^*\| < r/2$, provided $R > 0$ is large enough because a_e is uniformly exponentially decreasing. This gives the required operator norm estimate for this first term. Fix R. The next two terms (the second and third) on the right side can be manipulated into the correct form (as on the the right hand side of (6.4)) by applying Proposition 5.3 Part (a). For example in the third term Proposition 5.3 (a) applies to $J_{\mathcal{A}} J_e^* (1/h_\mu) \tilde{\chi}_{(-\infty,-2)}(x_1/R) h_\mu$ giving an expansion for the third term with the h_μ term commuted to the left in several terms and some other terms involving $\exp(-\nu\langle D_1\rangle)$. Part (b) can then be applied to these latter terms to commute the factor $\exp(-\nu\langle D_1\rangle)$ to the left as in (6.4). Part (b) applies because the symbol $a_{\mathcal{A}}$ is in $\mathcal{B}_{\mathcal{A}}$.

Therefore it remains only to consider the last term. Observe that this last term differs from $J_{\mathcal{A}}^*(1-JJ^*)^l V_2$ in that, the leftmost occurrence of the factor $1-JJ^*$ has been replaced by

$$A \equiv \tilde{\chi}_{(-\infty,-2)}(x_1/R)(1-J_{\mathcal{A}}J_{\mathcal{A}}^*)\tilde{\chi}_{(-\infty,-2)}(x_1/R).$$

This process of replacing factors of $1-JJ^*$ by A can be continued. Each step spawns terms that have an h_μ factor and Proposition 3 can be applied repeatedly to reduce those terms to the required form (as in (6.4)). In the end there is only one term that has not been reduced to the required form: it is $J_{\mathcal{A}}^* A^l V_2$. The symbols of $J_{\mathcal{A}}$, $\tilde{\chi}_{(-\infty,-2)}(x_1/R)J_{\mathcal{A}}$ and V_2 are in $\mathcal{B}_{\mathcal{A}}$ so that Proposition 5.3 can be applied repeatedly (Part (b) applies initially). This verifies the expansion (6.4) and the number of terms grows at most geometrically in l: $O(8^l)$ say.

The same reasoning applies to show that

$$J^*(1-JJ^*)^l h_\mu = \sum_{j\geq 1} \tilde{Q}_{4j}^* h_\mu \tilde{A}_{2j}\tilde{Q}_{4j+1} + e^{-\nu\langle D_1\rangle}\tilde{Q}_{4j+2}^*\tilde{A}_{2j+1}\tilde{Q}_{4j+3} \qquad (6.5)$$

where the $\tilde{Q}_k$ are in $\mathcal{B}_\delta$ and bounded (but not $O(\tilde{r})$ this time because there is no V_2 factor this time) and $\|\tilde{A}_k\| < C_\nu^{l+1} r^l$. Again there are $O(8^l)$ terms.

It is now possible to derive analytic extension results for operators involving $R_0(z)K^*$ from Proposition 1 by way of the expansions (6.4) and (6.5). More precisely let Q be an operator with symbol q in $\mathcal{B}$ and let $\mathcal{K}$ be a compact set intersecting both C_+ and C_-. Then,

$$h_\mu Q R_0(z) K^* V_2 \qquad (6.6)$$

$$(\text{resp.} \quad h_\mu Q R_0(z) K^* h_\mu) \qquad (6.7)$$

has an analytic extension to all of $\mathcal{K}$ as an operator on $L^2(\mathbf{R}^n)$ and there are constants $C>0$ and $m\in\mathbf{N}$ so that

$$\|h_\mu Q R_0(z) K^* V_2\| \;<\; C\|q\|_m \tilde{r} \qquad (6.8)$$

$$(\text{resp.} \; \|h_\mu Q R_0(z) K^* h_\mu\| \;<\; C\|q\|_m) \qquad (6.9)$$

for z in $\mathcal{K}$ with $\tilde{r}$ as in (6.2) and $\mu > 0$ as in the statement of the result. For if one replaces K^* in (6.6) by its partial Neumann sum and further substitutes in from (6.4) then the existence of an analytic extension follows from Proposition 1. It remains therefore to check that the partial sum converges uniformly on $\mathcal{K}$. The norm of the l-th term of the Neumann series is

$$\|h_\mu Q R_0(z) J^*(1 - JJ^*)^l V_2\| < C C_\nu^{l+1} 8^l \|q\|_m r^l \tilde{r},$$

by (6.4). Here r is as in (5.10). Provided $8C_\nu r < 1$ then the sum over l converges uniformly on $\mathcal{K}$ which shows that (6.6) has an analytic extension and it is bounded on $\mathcal{K}$ as claimed in (6.8). If one now further supposes that Q is a bounded operator from the domain $D(H_0)$ of H_0 (with the graph norm) to itself then precisely the same reasoning shows that the analytic extension of the expression (6.6) (resp. (6.7)) is bounded as a mapping from $L^2(\mathbf{R}^n)$ to $D(H_0)$

Return now to the case $Q \in \mathcal{B}$ is not necessarily bounded on $D(H_0)$ and suppose that $\{\Im z > -\nu\} \supseteq \mathcal{K}$ for some $\nu > 0$. Then

$$e^{-\nu\langle D_1\rangle} R_0(z) K^* V_2 \tag{6.10}$$

$$(\text{resp.} \quad e^{-\nu\langle D_1\rangle} R_0(z) K^* h_\mu) \tag{6.11}$$

has an analytic extension to all of $\mathcal{K}$ as an operator on $L^2(\mathbf{R}^n)$ and there are constants $C > 0$ and $m \in \mathbf{N}$ so that

$$\|e^{-\nu\langle D_1\rangle} Q R_0(z) K^* V_2\| \;<\; C\|q\|_m \tilde{r} \tag{6.12}$$

$$(\text{resp. } \|e^{-\nu\langle D_1\rangle} Q R_0(z) K^* h_\mu\| \;<\; C\|q\|_m) \tag{6.13}$$

for z in $\mathcal{K}$. The proof is almost identical to the proof that (6.6) has an analytic extension with bound (6.8). It is now possible to fix the choice of r in (5.10).

Recall the expansion (6.3) for $R(z)$. In order to establish the result for $h_\mu R(z) h_\mu$, multiply (6.3) on the left and right both by h_μ and expand both occurrences of $(1 + V_2 R_0(z) K^*)^{-1}$ as a Neumann series. (The series is convergent at least for $\Im z$ large enough.) One obtains,

$$h_\mu R(z) h_\mu [1 + \sum_{l\geq 0} A_{1,l}(z) + V_0 A_{2,l}(z)] = \sum_{l\geq 0} B_l(z) \tag{6.14}$$

where

$$B_l(z) = h_\mu J R_0(z) K^* [V_2 R_0(z) K^*]^l h_\mu \tag{6.15}$$

and, for $j = 1, 2$

$$A_{j,l}(z) = h_\mu Q_j R_0(z) K^* [V_2 R_0(z) K^*]^l h_\mu \tag{6.16}$$

where $Q_1 = h_{\mu_0} h_\mu^{-2} V_1$ and $Q_2 = h_{\mu_0} h_\mu^{-2} J$ so that $Q_1, Q_2 \in \mathcal{B}$. (It is possible to assume $2\mu \leq \mu_0$ without loss of generality.)

It will be shown that $A_{1,l}(z)$, $A_{2,l}(z)$ and $B_l(z)$ extend analytically in z to all of $\mathcal{K} \subseteq \mathbf{C}$ as bounded operators on $L^2(\mathbf{R}^n)$ and $A_{2,l}(z)$ extends in fact as an operator from $L^2(\mathbf{R}^n)$ to $D(H_0)$. Moreover it will be shown that there is a constant $C_0 > 0$ so that

$$\|A_{1,l}(z)\| + \|(H_0 + i) A_{2,l}(z)\| + \|B_l(z)\| < C_0^{l+1} (\tilde{r})^l \tag{6.17}$$

for $z \in \mathcal{K}$. (The extensions are also denoted by the same symbols.) Provided $\tilde{r} > 0$ of (6.2) is chosen so that $C_0 \tilde{r} < 1$ then the series of extended functions in (6.14) converges. If it is further shown that $A_{1,l}(z) + V_0 A_{2,l}(z)$ is compact then the result follows for $h_\mu R(z) h_\mu$ by the analytic Fredholm theorem [20, Equation VII.1.9].

Check the compactness first. The symbol $t_1(x, \xi)$ of V_1 goes to 0 along with all its derivatives as $|(x, \xi)| \to \infty$. (x and ξ can be restricted to be real here.) Therefore V_1 is the operator norm limit (see (5.8)) of integral operators with compactly supported symbols (and hence Schmidt class [20]) and is therefore compact. As for $V_0 A_{2,l}(z)$, since V_0 is H_0-compact, it suffices to check that J is bounded from the domain of H_0 (with the graph norm) to itself, for recall the definition Q_2 in equation (6.16). But

$$J R_0(i) - (H_\mathcal{A} + i)^{-1} J = (H_\mathcal{A} + i)^{-1} [H_\mathcal{A} J - J H_0] R_0(i)$$

where $H_\mathcal{A} = H_0 + V_\mathcal{A}$ has the same domain as H_0 and $H_\mathcal{A} J - J H_0$ is bounded by Proposition 3.1 and so this is obvious.

Check next that $A_{j,l}$ and B_l, $l \in \mathbf{N}_0$ $j = 1, 2$ have analytic extensions. This is just a matter of expressing each of them as a product of operators of the four forms, (6.6), (6.7), (6.10) and (6.11) and bounded operators that don't depend on z. There is no difficulty here provided one notes that the symbol of V_2 has compact support in the ξ_1 variable so that $V_2 = V_2[\chi_{(-R,R)}(D_1) e^{\nu\langle D_1 \rangle}] e^{-\nu\langle D_1 \rangle}$ for R large enough and the operator in brackets is bounded. When considering $A_{2,l}$ we get the stronger conclusion that the extension is bounded as an operator from $L^2(\mathbf{R}^n)$ to $D(H_0)$ because J is bounded as operator from $D(H_0)$ to itself. The operator bound

(6.17) follows from the bounds for the factors (6.8),(6.9) (6.12)(6.13). This completes the proof of the theorem for $h_\mu R(z) h_\mu$

The proof for $V_2^* R(z) h_\mu$ is very similar except for one detail. Instead of B_l of the preceding argument one obtains

$$\tilde{B}_l(z) = V_2^* J R_0(z) K^* [V_2 R_0(z) K^*]^l h_\mu.$$

If one replaces $J = J_e + J_{\mathcal{A}}$ in the expansion of $\tilde{B}_l(z)$ by J_e then the argument above for B_l applies. However, when $J_{\mathcal{A}}$ replaces J, then one additional step is required to take advantage of the compact support in ξ_1 of $t_2(x, \xi)$ and that is to apply Proposition 5.3 Part (b). Then the remainder of the argument for $\tilde{B}_l(z)$ is much like that for $B_l(z)$. Again $\tilde{B}_l$ has an analytic extension and $\|\tilde{B}_l\| < C_0^{l+1} (\tilde{r})^l$.

It remains to consider $h_\mu R(z) V_2$ and $V_2^* R(z) V_2$. The difference between these two cases is much like the difference between the preceding two cases and so it suffices to consider $h_\mu R(z) V_2$. Multiply through equation (6.1) by V_2 on the right and h_μ on the left:

$$\begin{aligned} & h_\mu R(z) V_2 [1 + \tilde{\chi}_{(-R,R)}(D_1) R_0(z) K^* V_2] \\ & \quad = \; -h_\mu R(z) h_{\mu_0} [V_1 + V_0 J] R_0(z) K^* V_2 + h_\mu J R_0(z) K^* V_2 \end{aligned} \qquad (6.18)$$

where $\tilde{\chi}_{(-R,R)}$ is in $C_0^\infty(\mathbf{R})$ and $\tilde{\chi}_{(-R,R)}(\xi_1) = 1$ for ξ_1 in the support of t_2. By the work already done, it is clear that the right side has a meromorphic extension to all of $\mathcal{K}$. Moreover

$$\tilde{\chi}_{(-R,R)}(D_1) R_0(z) K^* V_2$$

has an analytic extension, being of the form (6.10), and it's operator norm is less than 1, provided that $z \in \mathcal{K}$ and $\tilde{r}$ in (6.2) is chosen adequately small. Fix $\tilde{r}$. Therefore it is possible to solve for $h_\mu R(z) V_2$ in (6.18) and this shows it is meromorphic. Because $\nu > 0$ and the compact set $\mathcal{K}$ were arbitrary, the result follows. □

7 Representation of the Scattering Matrix.

Theorem 1.1 is derived in this Section along with the representation (1.8) there for the scattering matrix. This is a standard representation as appeared in Kuroda's paper [25, 1973], adapted to the two Hilbert space setting as in Isozaki and Kitada's work [19, 1985] and further adapted to the Stark effect.

In preparation for the proof recall that the two Hilbert space wave operators $W_J^{\pm}$ of (1.1) are known to exist and be complete (by the argument in [32] (where $a = 1$)). In addition it can be shown that, for any $u \in \mathcal{S}(\mathbf{R}^n)$

$$\int_{-\infty}^{\infty} \|V_J e^{-itH_0} u\| dt < \infty \tag{7.1}$$

(which implies existence of $W_J^{\pm}$ by Cook's argument [7]). This is verified in [32] but a more direct proof can be outlined as follows. Recall the Avron-Herbst formula [3]

$$e^{-itH_0} = e^{-iF^2t^3/3} e^{-iFtx_1} e^{iFt^2D_1 - it(-\Delta)}$$

and observe that $\exp(-itFx_1)$ acts by translation by tF units in the momentum ξ_1 variable (and similarly $\exp(-iFt^2D_1)$ translates in the x_1 variable.) Since $V_J = h_{\mu_0}(V_0J + V_1) + V_2$ (see (6.2)) it suffices to verify (7.1) when V_J is replaced by V_2 and by $h_{\mu_0}(V_0J + V_1)$. The former case is easily settled because the symbol of V_2 is compactly supported in ξ_1 and so a translation argument suffices. For the latter case when V_J is replaced by $h_{\mu_0}(V_0J + V_1)$, it is further possible to replace V_J by $[\tilde{\chi}_{(-\infty,0)} + \langle x_1\rangle^{-k}\tilde{\chi}_{(-\infty,0)}]$ for some $k > 1$ in view of Lemma 5.1. In this case the argument is a relatively standard stationary phase argument (as in [14, pp 70-72] for example) except that the Fourier transform of u need not compactly supported and so a partition of unity argument is needed as well. With this preparation it is possible to prove Theorem 1.1.

Proof of Theorem 1.1: Begin by showing that the right hand side of the equation (1.8) extends to a meromorphic function taking values in the space of bounded operators on $L^2(\mathbf{R}^{n-1})$. Consider the first term. Since $V_J = h_{\mu_0}(V_1 + V_0J) + V_2$ and $J = J_e + J_{\mathcal{A}}$ in the notation of the preceding Section, the first term is (omitting the factor $-2\pi i$)

$$\begin{aligned} T_0(\lambda)J^*V_JT_0(\lambda)^* &= T_0(\lambda)J^*gh_\mu h_\mu V_1T_0(\lambda)^* \\ &\quad + T_0(\lambda)J^*gh_\mu[V_0R_0(i)](H_0 - i)h_\mu JT_0(\lambda)^* \\ &\quad + T_0(\lambda)J_e^*V_2T_0(\lambda)^* + T_0(\lambda)J_{\mathcal{A}}^*V_2T_0(\lambda)^*. \end{aligned} \tag{7.2}$$

where $g \in C_b^\infty$ is defined by $h_{\mu_0} = gh_\mu^2$ and $2\mu \leq \mu_0$. The first three terms on the right hand side of (7.2) are entire by Proposition 5.2. (Note g can be incorporated into the symbol of J^* for the application of Proposition 5.2.) Proposition 5.2 does not immediately apply to the fourth term on the right hand side of (7.2), but Proposition 5.3 (b) does apply because $J_{\mathcal{A}}$ and V_2 are in $\mathcal{B}_{\mathcal{A}}$ and it brings the second term into the correct form to apply Proposition 5.2.

Next claim that the term $T_0(\lambda)V_J^*R(\lambda+i0)V_JT_0(\lambda)^*$ in (1.8) has a meromorphic extension. As above, it is convenient to expand V_J as

$$V_J = h_\mu h_\mu g V_1 + h_\mu[gV_0R_0(i)](H_0-i)h_\mu J + V_2$$

and similarly for V_J^*. Then it is simply a matter of applying Theorem 6.2 and Proposition 5.2. Thus the right hand side of (1.8) has a meromorphic extension.

It remains to establish (1.8). The argument here is similar to that of Isozaki and Kitada's [19, Theorem 3.3]. Begin by considering the L^2 inner product ((S-1)u,v) for u and v in fundamental subsets of $L^2(\mathbf{R}^n)$. (A subset is *fundamental* if its linear span is dense.) Specifically, suppose that $v \in \mathcal{S}(\mathbf{R}^n)$ and for some bounded real interval (a,b), $v = E_0(a,b)v$ where E_0 is the spectral measure of H_0. That all such states form a fundamental subset follows by considering the set's image under the spectral representation U. As for u, let u_1 be in the same fundamental subset as v and for some $m = 1, 2, \ldots$, let $u = e^{-\langle D_1\rangle/m}u_1$.

Expand $S-1 = (W_J^+)^*(W_J^- - W_J^+)$. (Recall $W_J^\pm$ are isometries [32, Theorem 2.1].) On the other hand,

$$[W_J^\pm - J]u = i\int_0^{\pm\infty} e^{itH}V_Je^{-itH_0}u\,dt$$

(for integrate and differentiate). The integral exists absolutely according to the opening remarks of this Section. By the Intertwining Principle [30]

$$\begin{aligned}((S-1)u,v) &= -i\int_{-\infty}^{\infty}(V_Je^{-itH_0}u, W_J^+e^{-itH_0}v)\,dt \\ &= T_1+T_2\end{aligned}$$

where

$$\begin{aligned}T_1 &= -i\int_{-\infty}^{\infty}(V_Je^{-itH_0}u, Je^{-itH_0}v)\,dt \\ T_2 &= -\int_{-\infty}^{\infty}\int_0^{\infty}(V_Je^{-itH_0}u, e^{isH}V_Je^{-i(s+t)H_0}v)\,ds\,dt.\end{aligned}$$

Consider first T_2:

$$\begin{aligned} T_2 &= -\lim_{\sigma,\tau\to 0+}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\sigma s}e^{-\tau|t|}(V_J^* e^{-isH}V_J e^{-itH_0}u, e^{-i(s+t)H_0}v)\,ds\,dt \\ &= -\lim_{\sigma,\tau\to 0+}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\sigma s}e^{-\tau|t|} \\ &\qquad \int_a^b (T_0(\lambda)V_J^* e^{-is(H-\lambda)}V_J e^{-it(H_0-\lambda)}u, T_0(\lambda)v)\,d\lambda\,ds\,dt \end{aligned}$$

by (5.4). (Recall $T_0(\lambda)V_J^*$ is bounded by Lemma 5.1.) Recall the identity

$$\int_0^{\infty} e^{-\sigma s}e^{-is(H-\lambda)}\,ds = -iR(\lambda+i\sigma)$$

in the strong operator topology (which is comparable to a Laplace transform; see [20, §9.1.3] for example). Therefore

$$T_2 = \lim_{\sigma,\tau\to 0+}\int_a^b (T_0(\lambda)V_J^* R(\lambda+i\sigma)V_J[R_0(\lambda+i\tau)-R_0(\lambda-i\tau)]u, T_0(\lambda)v)\,d\lambda.$$

It is possible to evaluate the limit $\tau\to 0+$ by applying Proposition 6.1. It is for this application, that the choice of $u = e^{-\langle D_1\rangle/m}u_1$ is convenient. (The conclusion of Proposition 6.1 is valid when either Q_j, $j=1$ or 2, there is replaced by the identity operator; the proof is very similar.) The limit as $\sigma\to 0+$ exists according to Theorem 6.2 so that

$$T_2 = 2\pi i\int_a^b (T_0(\lambda)V_J^* R(\lambda+i0)V_J T_0(\lambda)^* T_0(\lambda)u, T_0(\lambda)v)\,d\lambda$$

Thus T_2 corresponds to the second term on the right side of (1.8).

Similar reasoning shows that T_1 is

$$\begin{aligned} T_1 &= -\lim_{\tau\to 0+}\int_a^b (T_0(\lambda)J^*V_J[R_0(\lambda+i\tau)-R_0(\lambda-i\tau)]u, T_0(\lambda)v)\,d\lambda \\ &= -2\pi i\int_a^b (T_0(\lambda)J^*V_J T_0(\lambda)^* T_0(\lambda)u, T_0(\lambda)v)\,d\lambda \end{aligned}$$

so that T_1 corresponds to the first term on the right side of of (1.8). Equation (1.8) follows from the uniqueness of the Radon Nikodym derivative almost everywhere and because the u, v belong to a fundamental set. □

A Appendix

Proofs of Lemmas 2.2, 3.2 and 3.3 are given.

Proof of Lemma 2.2. Let I denote the integral expression in the statement of the Lemma. Neglecting those terms in the integrand which are linear in t, one arrives at the estimate:

$$I \leq a^{-k+1/2} c^{-j} \int_0^\infty (1+t^2)^{-k} (1+\frac{a}{c} t^2)^{-j} \, dt$$

after a change of variable. Now assume $a/c \geq 1$:

$$I \leq a^{-k+1/2} c^{-j} \int_0^\infty (1+t^2)^{-k-j} \, dt$$

The integral expression on the right side can be expressed in terms of the Beta function [1, p. 258] (after a trigonometric substitution): it is $B(1/2, k+j-1/2)/2$. It is known that $\sqrt{l} B(1/2, l)$ is bounded uniformly on any interval of the form $l > \epsilon$. (for expand B in terms of gamma functions and apply Stirling's formula). This implies the present result in the case $a \geq c$. Interchanging the roles of a and c, k and j, a symmetric argument completes the proof of part (a).

It remains to consider part (b) when $a < c$ so that $b > \sqrt{c}$, because $c = (a+b^2)/2$. If one ignores the terms in the integrand of I which are quadratic in t,

$$\begin{aligned} I &\leq c^{-j/2} a^{-k+1} (2b)^{-1} \int_0^\infty (1+\frac{a}{c} t)^{-j/2} (1+t)^{-k} dt \\ &\leq c^{-j/2} a^{-k+1} (2b)^{-1} (k-1)^{-1} \end{aligned}$$

after a change of variable. This proves part (b), since $b > \sqrt{c}$. □

Proof of Lemma 3.2. From the expression (3.9) for $x(t)$

$$|\Re x_1(t)| \geq -\Re y_1 - 2\Re \eta_1 t + Ft^2 - C_\theta t$$

for some constant $C_\theta > 0$ by (2.7). Therefore, in the case that $-\Re\eta_1 > C_\theta/2$ the present Lemma follows from Lemma 2.2 (with $b = \Re\eta_1 - C_\theta/2$ there). It is possible to assume therefore that $|\Re\eta_1|$ is bounded. In this case a shift of variables argument in the integral brings it into the correct form to apply Lemma 2.2 and derive the required estimates. □

Proof of Lemma 3.3. In view of Lemma 3.2 it suffices to show that

$$\lim_{|(y_\perp,\eta_\perp)|\to\infty} \int_0^\infty q(x(t,y,\eta),\xi(t,\eta))\,dt = 0$$

(because $j' < j$.) The tail of the integral is, for any $h > 0$

$$\int_h^\infty q(x(t,y,\eta),\xi(t,\eta))\,dt = \int_0^\infty q(x(t,y',\eta'),\xi(t,\eta'))\,dt$$

where $y' = x(h,y,\eta)$ and $\eta' = \xi(h,\eta)$ since $x(t)$ and $\xi(t)$ are solutions of an autonomous system. Therefore Lemma 3.2 implies that this tail can be made small by choosing h large and it is small uniformly in $(y_\perp,\eta_\perp)$. Fix h. It remains to show

$$\lim_{|(y_\perp,\eta_\perp)|\to\infty} \int_0^1 q(x(th,y,\eta),\xi(th,\eta))\,dt = 0. \tag{A.1}$$

Distinguish two cases: First, the case $\eta_\perp$ is small, say $|\eta_\perp| < |y_\perp|/3h$. Then $|(x_\perp(th,y,\eta),\eta_\perp(th,\eta))|$ gets large as $|(y_\perp,\eta_\perp)|$ does by equation (3.9) and so (A.1) follows simply because $q(x,\xi)\to 0$ as $|(x_\perp,\xi_\perp)|\to\infty$. In the second case, when $|\eta_\perp|$ is large, $x_\perp(th,y,\eta)$ cannot stay within any bounded region for more than a very short time: for $0 \le \tau \le t \le 1$

$$|x_\perp(th,y,\eta) - x_\perp(\tau h,y,\eta)| \ge h|t-\tau|(2|\eta_\perp| - C)$$

for some constant C by (3.9). This implies (A.1) in this case as well. □

Acknowledgement. This work is partly funded by a URAFP award from the University of Toledo. The author wishes to thank the University of Kentucky for hospitality at the outset of the project and especially Peter Hislop. It is also a pleasure to thank Professor Alan Carey for his hospitality during visits to the University of Adelaide.

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